SASAKI-EINSTEIN AND PARASASAKI-EINSTEIN METRICS FROM (κ, μ)-STRUCTURES

B. CAPPELLETTI MONTANO, A. CARRIAZO, AND V. MARTÍN-MOLINA

ABSTRACT. We prove that any non-Sasakian contact metric (κ, μ)-space admits a canonical η-Einstein Sasakian or η-Einstein paraSasakian metric. An explicit expression for the curvature tensor fields of those metrics is given and we find the values of κ and μ for which such metrics are Sasaki-Einstein and paraSasaki-Einstein. Conversely, we prove that, under some natural assumptions, a K-contact or K-paracontact manifold foliated by two mutually orthogonal, totally geodesic Legendre foliations admits a contact metric (κ, μ)-structure. Furthermore, we apply the above results to the geometry of tangent sphere bundles and we discuss some topological and geometrical properties of (κ, μ)-spaces related to the existence of Eistein-Weil and Lorentzian Sasakian Einstein structures.

1. INTRODUCTION

It is well known that the tangent sphere bundle $T_1 N$ of a flat Riemannian manifold N carries a contact Riemannian structure such that $R(X, Y)\xi = 0$ for any vector fields X, Y on $T_1 N$, where the Reeb vector field ξ is given by twice the geodesic flow. The class of contact metric manifolds satisfying the above condition, which were at first studied by Blair in [5], is not preserved by $\mathcal{D}$-homothetic transformations. In fact, if one deforms $\mathcal{D}$-homothetically the structure, one falls in the larger class of "contact metric (κ, μ)-spaces", i.e. contact metric manifolds (M, ϕ, ξ, η, g) satisfying

$$(1.1) \qquad R(X, Y)\xi = \kappa \left(\eta\left(Y\right) X - \eta\left(X\right) Y \right) + \mu \left(\eta\left(Y\right) hX - \eta\left(X\right) hY \right),$$

for some constants κ and μ, where $2h$ denotes the Lie derivative of the structure tensor φ in the direction of the Reeb vector field (see § 2 for more details). This new class of Riemannian manifolds was introduced in [8] as a natural generalization both of the contact metric manifolds such that $R(X, Y)\xi = 0$ and of the Sasakian condition $R(X, Y)\xi = \eta\left(Y\right) X - \eta\left(X\right) Y$. Despite the technical appearance of the definition, nowadays contact (κ, μ)-spaces are considered an important topic in contact Riemannian geometry because there are good reasons for studying them. The first is that, while the values κ and μ vary, one proves that the condition (1.1) remains unchanged under $\mathcal{D}$-homothetic deformations. Next, in the non-Sasakian case (that is for $\kappa \neq 1$), the condition (1.1) determines the curvature tensor field completely. Furthermore, (κ, μ)-spaces provide non-trivial examples of some remarkable classes of contact Riemannian manifolds, like CR-integrable contact metric manifolds ([35]), H-contact manifolds ([32]) and harmonic contact metric manifolds ([37]). Finally, there are non-trivial examples of such Riemannian manifolds, the most important being the tangent sphere bundle of any Riemannian manifold of constant sectional curvature with its standard contact metric structure.

In this paper we study the strict relations between the theory of (κ, μ)-spaces and other two important topics of contact geometry, namely Sasakian and paraSasakian manifolds. In fact, given a non-Sasakian (κ, μ)-space $(M, \varphi, \xi, \eta, g)$ we describe a method for constructing a Sasakian or paraSasakian metric on M compatible with the same contact form η. The type of metric (Sasakian or paraSasakian) depends on the value of a well-known invariant introduced by Boeckx in [9] in order to classify (κ, μ)-spaces, defined as

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1 - \kappa}}.$$

More precisely, we are able to define a Sasakian or paraSasakian metric if $|I_M| > 1$ or $|I_M| < 1$, respectively. Moreover, by using the aforementioned property that the (κ, μ)-nullity condition (1.1) determines the curvature completely, we find an explicit expression for the curvature tensor field of the above Sasakian and paraSasakian metrics. We obtain from it our main result, that such metrics are always η-Einstein and that for some values of κ and μ they are Sasaki-Einstein and paraSasaki-Einstein, though the starting (κ, μ)-structure can never be Einstein in dimension greater than 3 ([7, p. 131]).

In the case $|I_M| > 1$ such a canonical Sasakian metric is positive or negative according to the sign of I_M, and by applying a result of Tanno on positive η-Einstein Sasakian manifolds, it turns out that every contact metric (κ, μ)-space such that $I_M > 1$ admits a Sasaki-Einstein metric. Furthermore, we prove that in dimension greater than or equal to 5, every (κ, μ)-space such that $I_M > 1$ carries also an Einstein-Weil structure.

On the other hand, in the case $I_M < -1$ we prove that M always admits a Lorentzian Sasakian-Einstein structure, that is a Lorentzian metric of signature $(1, 2n)$ such that the Reeb vector field ξ is time-like and the cone $(M \times \mathbb{R}_+, -dt^2 + r^2 g)$ is pseudo-Calabi-Yau with Einstein constant $-2n$. This notion arose in the context of the theory of Killing spinors in Lorentzian geometry (cf. [4], [11]). Recently, non-trivial examples of Lorentzian Sasakian-Einstein metrics have been found on the connected sums of $S^2 \times S^3$ ([23]).

We then discuss some consequences of such results on the geometry of tangent sphere bundles, which are standard examples of (κ, μ)-spaces, and more precisely on the problem of finding η-Einstein metrics on them (recently considered in [1], [10], [21], [31], among others). We prove that if N is a space form of constant sectional curvature $c \neq 1$ of dimension $n + 1$, then in the case $c > 0$ the tangent sphere bundle $T_1 N$ admits an η-Einstein Sasakian metric $\overline{g}$, whose Ricci tensor is given by

$$\overline{\mathrm{Ric}} = 2(2n\sqrt{c} - 1)\overline{g} + 2(-2n\sqrt{c} + n + 1)\eta \otimes \eta,$$

where η is the standard contact form on $T_1 N$ determined by twice the geodesic flow. It follows that if

(1.2)
$$c = \frac{1}{4}\left(1 + \frac{1}{n}\right)^2$$

and $\dim(N) > 2$ then $\overline{g}$ is Sasaki-Einstein. By applying a suitable $\mathcal{D}$-homothetic deformation, it follows from the above result that in dimension greater than or equal to 5 the tangent sphere bundle of every space form of positive sectional curvature carries a Sasaki-Einstein metric. This extends the result of Tanno that $T_1 S^3 \simeq S^2 \times S^3$ carries a Sasaki-Einstein metric (more recently, other Sasaki-Einstein metrics have been defined on $S^2 \times S^3$, see [12, Chapter 11] and references therein).

The right-hand side of (1.2) plays a role also in the context of Weil structures. In fact, we prove that $T_1 N$ admits both the Einstein-Weil structure $W^+ = (\overline{g}', \theta)$ and $W^- = (\overline{g}', -\theta)$, for some 1-form θ, where $\overline{g}'$ is a Sasakian metric $\mathcal{D}$-homothetic to $\overline{g}$, and if c is strictly greater than the right-hand side of (1.2) then $\overline{g}'$ belongs to the conformal class of $\overline{g}$.

Meanwhile, in the case $c < 0$, $T_1 N$ admits an η-Einstein paraSasakian metric $\widetilde{g}$ with Ricci tensor

$$\widetilde{\mathrm{Ric}} = (-4n\sqrt{-c} + 3)\widetilde{g} + (4n\sqrt{-c} - 2n - 3)\eta \otimes \eta,$$

which is Einstein if and only if

$$c = -\frac{1}{16}\left(2 + \frac{3}{n}\right)^2.$$

Again by applying a suitable $\mathcal{D}$-homothetic deformation, it follows that, provided that $c \neq -\frac{1}{16n^2}$, T_1N carries a paraSasakian-Einstein metric. Moreover, by using a result of Blair ([6]), such paraSasakian structures turn out to be never regular, provided that N is compact.

To the knowledge of the authors, the above paraSasakian structures appear to provide the first non-trivial examples of η-Einstein (eventually Einstein) paraSasakian manifolds. In fact, while there has been an increasing interest in last years in paraSasakian geometry (see [3], [25], [38]), so far the only known examples of Einstein paracontact manifolds seem to be the hyperboloid $\mathbb{H}_{n+1}^{2n+1}(1)$ of constant curvature -1 ([25]) and $\mathbb{R}_1^3$ with the flat metric ([39]). Moreover no explicit example of η-Einstein paraSasakian manifold were known.

Finally, in the last part of the paper we will give a geometric interpretation to the above canonical Sasakian and paraSasakian metrics. It is well known that any non-Sasakian (κ, μ)-space is foliated by two Legendre foliations, defined by the eigendistributions of the operator h, and that such a foliated structure plays an important role in the theory of (κ, μ)-spaces (cf. [16], [19]). We show that the geometry of these Legendre foliations, encoded by some invariants like the Pang invariant ([30]) and the Libermann map ([27]), is fully described by the above Sasakian and paraSasakian metrics. In this way we are able to find a sufficient condition for a K-contact (respectively, K-paracontact) manifold M, foliated by two mutually orthogonal, totally geodesic Legendre foliations, to admit a contact metric (κ, μ)-structure, compatible with the same underlying contact form, such that $|I_M| > 1$ (respectively, $|I_M| < 1$).

2. Preliminaries

2.1. Contact metric (κ, μ)-spaces.
An *almost contact structure* on a $(2n+1)$-dimensional smooth manifold M is simply a triplet (φ, ξ, η), where φ is a tensor field of type $(1,1)$, η a 1-form and ξ a vector field on M satisfying the following conditions

$$(2.1) \qquad \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where I is the identity mapping. From (2.1) it follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and the $(1,1)$-tensor field φ has constant rank $2n$ ([7]). Given an almost contact manifold (M, φ, ξ, η) one can define an almost complex structure J on the product $M \times \mathbb{R}$ by setting $J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta\left(X\right)\frac{d}{dt}\right)$ for any $X \in \Gamma\left(TM\right)$ and $f \in C^\infty\left(M \times \mathbb{R}\right)$. Then the almost contact manifold is said to be *normal* if the almost complex structure J is integrable. The condition for normality is given by the vanishing of the tensor field $N_\varphi := [\varphi, \varphi] + 2d\eta \otimes \xi$.

Any almost contact manifold (M, φ, ξ, η) admits a *compatible metric*, i.e. a Riemannian metric g satisfying

$$(2.2) \qquad g\left(\varphi X, \varphi Y\right) = g\left(X, Y\right) - \eta\left(X\right)\eta\left(Y\right)$$

for all $X, Y \in \Gamma\left(TM\right)$. The manifold M is said to be an *almost contact metric manifold* with structure (φ, ξ, η, g). From (2.2) it follows immediately that $\eta = g(\cdot, \xi)$ and $g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot)$. Then one defines the 2-form Φ on M by $\Phi\left(X, Y\right) = g\left(X, \varphi Y\right)$, called the *fundamental* 2-*form* of the almost contact metric manifold. If $\Phi = d\eta$ then η becomes a contact form, with ξ its corresponding Reeb vector field, and $(M, \varphi, \xi, \eta, g)$ is called a *contact metric manifold*.

In a contact metric manifold one has

$$(2.3) \qquad \nabla\xi = -\varphi - \varphi h$$

where ∇ is the Levi-Civita connection of (M, g) and h denotes the $(1,1)$-tensor field defined by $h := \frac{1}{2}\mathcal{L}_\xi\varphi$. The tensor field h is symmetric with respect to g and vanishes identically if and only if the Reeb vector field ξ is Killing. In this last case the contact metric manifold is said to be *K-contact*. A normal contact

metric manifold is called a *Sasakian manifold*. Any Sasakian manifold is K-contact and the converse holds in dimension 3.

If the Ricci tensor of a contact metric manifold has the following form

$$\text{Ric} = ag + b\eta \otimes \eta \tag{2.4}$$

for some functions a and b, we say that M is η-Einstein. It is known that if M is Sasakian and $\dim(M) \geq 5$, then a and b are necessarily constants. This notion appears to be a good generalization of the concept of Einstein metrics in the context of contact Riemannian geometry. Many interesting geometric properties of η-Einstein metrics are presented in the recent paper [13].

Given a positive constant $c > 0$, a $\mathcal{D}_c$-*homothetic deformation* on a contact metric manifold $(M, \varphi, \xi, \eta, g)$ is the change of the structure tensors of the following type

$$\varphi' := \varphi, \quad \xi' := \frac{1}{c}\xi, \quad \eta' := c\eta, \quad g' := cg + c(c-1)\eta \otimes \eta. \tag{2.5}$$

Then $(\varphi', \xi', \eta', g')$ is again a contact metric structure on M.

A recent generalization of Sasakian manifolds is the notion of *contact metric* (κ, μ)-*space* ([8]). Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold. If the curvature tensor field of the Levi-Civita connection satisfies

$$R(X,Y)\xi = \kappa\left(\eta\left(Y\right)X - \eta\left(X\right)Y\right) + \mu\left(\eta\left(Y\right)hX - \eta\left(X\right)hY\right), \tag{2.6}$$

for some $\kappa, \mu \in \mathbb{R}$, we say that $(M, \varphi, \xi, \eta, g)$ is a *contact metric* (κ, μ)-*space* (or that ξ belongs to the (κ, μ)-nullity distribution). This definition was introduced and deeply studied by Blair, Koufogiorgos and Papantoniou in [8]. Among other things, the authors proved the following result.

Theorem 2.1 ([8]). *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ)-space. Then*

$$h^2 = -(1-\kappa)\varphi^2, \tag{2.7}$$

so that necessarily $\kappa \leq 1$. Moreover, if $\kappa = 1$ then $h = 0$ and $(M, \varphi, \xi, \eta, g)$ is Sasakian. If $\kappa < 1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions $\mathcal{D}_h(0) = \mathbb{R}\xi$, $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ given by the eigenspaces of h corresponding to the eigenvalues 0, λ and $-\lambda$, where $\lambda = \sqrt{1-\kappa}$.

The same authors also proved the following formulas for the covariant derivatives of the tensor fields φ, h and φh ([8]):

$$(\nabla_X \varphi)Y = g\left(X + hX, Y\right)\xi - \eta\left(Y\right)\left(X + hX\right), \tag{2.8}$$

$$(\nabla_X h)Y = ((1-\kappa)g(X, \varphi Y) - g(X, \varphi hY))\xi + \eta(Y)h(\varphi X + \varphi hX) - \mu\eta(X)\varphi hY, \tag{2.9}$$

$$(\nabla_X \varphi h)Y = (g(X, hY) - (1-\kappa)g(X, \varphi^2 Y))\xi + \eta(Y)(hX - (1-\kappa)\varphi^2 X) + \mu\eta(X)hY. \tag{2.10}$$

Notice that while $\mathcal{D}$-homothetic deformations preserve the state of being contact metric, K-contact, Sasakian or η-Einstein, they destroy conditions like $R(X,Y) = 0$ or $R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y)$. However, they preserve the class of contact metric (κ, μ)-structures. Indeed, if (φ, ξ, η, g) is a (κ, μ)-structure then the deformed structure $(\varphi', \xi', \eta', g')$ is a (κ', μ')-structure with

$$\kappa' = \frac{\kappa + c^2 - 1}{c^2}, \quad \mu' = \frac{\mu + 2c - 2}{c}. \tag{2.11}$$

In [9] Boeckx provided a local classification of non-Sasakian (κ, μ)-space based on the number

$$I_M := \frac{1 - \frac{\mu}{2}}{\sqrt{1-\kappa}},$$

which is an invariant of a contact metric (κ, μ)-structure up to $\mathcal{D}$-homothetic deformations. He proved that two non-Sasakian contact metric (κ, μ)-spaces $(M_1, \varphi_1, \xi_1, \eta_1, g_1)$ and $(M_2, \varphi_2, \xi_2, \eta_2, g_2)$ are locally

isometric as contact metric manifolds, up to $\mathcal{D}$-homothetic deformations, if and only if $I_{M_1} = I_{M_2}$. A geometric interpretation of the invariant I_M and of the Boeckx's classification was recently given in [16].

The standard example of contact metric (κ, μ)-spaces is given by the tangent sphere bundle $T_1 N$ of a manifold of constant curvature $c \neq 1$ endowed with its standard contact metric structure. In this case $\kappa = c(2 - c)$, $\mu = -2c$ and $I_{T_1 N} = \frac{1+c}{|1-c|}$. Other examples are given by certain Lie groups defined by Boeckx in [9].

We conclude the subsection by recalling the following formula for the Lie derivative of the operator h in any non-Sasakian (κ, μ)-space (cf. [20, Lemma 4.5])

$$(2.12) \qquad \mathcal{L}_\xi h = (2 - \mu)\varphi h + 2(1 - \kappa)\varphi.$$

2.2. **Paracontact geometry.** An *almost paracontact structure* (cf. [26]) on a $(2n + 1)$-dimensional smooth manifold M is given by a $(1,1)$-tensor field $\widetilde{\varphi}$, a vector field ξ and a 1-form η satisfying the following conditions

   (i) $\eta(\xi) = 1$, $\widetilde{\varphi}^2 = I - \eta \otimes \xi$,
   (ii) the eigendistributions $\mathcal{D}^+$ and $\mathcal{D}^-$ of $\widetilde{\varphi}$ corresponding to the eigenvalues 1 and -1, respectively, have equal dimension n.

As an immediate consequence of the definition, one has that $\widetilde{\varphi}\xi = 0$, $\eta \circ \widetilde{\varphi} = 0$ and the field of endomorphisms $\widetilde{\varphi}$ has constant rank $2n$. As for the almost contact case, one can consider the almost paracomplex structure on $M \times \mathbb{R}$ defined by $\widetilde{J}\left(X, f\frac{d}{dt}\right) = \left(\widetilde{\varphi}X + f\xi, \eta(X)\frac{d}{dt}\right)$, where X is a vector field on M and f a C^∞ function on $M \times \mathbb{R}$. By definition, if $\widetilde{J}$ is integrable the almost paracontact structure $(\widetilde{\varphi}, \xi, \eta)$ is said to be *normal*. The computation of $\widetilde{J}$ in terms of the tensors of the almost paracontact structure leads us to define a tensor field $N_{\widetilde{\varphi}}$ of type $(1,2)$ given by $N_{\widetilde{\varphi}} := [\widetilde{\varphi}, \widetilde{\varphi}] - 2d\eta \otimes \xi$. The almost paracontact structure is then normal if and only if $N_{\widetilde{\varphi}}$ vanishes identically (cf. [38]). Normality in paracontact geometry has the following geometric interpretation.

Theorem 2.2 ([15]). *An almost paracontact manifold $(M, \widetilde{\varphi}, \xi, \eta)$ is normal if and only if the eigendistributions $\mathcal{D}^+$ and $\mathcal{D}^-$ of $\widetilde{\varphi}$ corresponding to the eigenvalues ± 1 are integrable and the vector field ξ is a foliated vector field with respect to both the foliations.*

If an almost paracontact manifold is endowed with a semi-Riemannian metric $\widetilde{g}$ such that

$$(2.13) \qquad \widetilde{g}(\widetilde{\varphi}X, \widetilde{\varphi}Y) = -\widetilde{g}(X, Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(TM)$, then $(M, \widetilde{\varphi}, \xi, \eta, \widetilde{g})$ is called an *almost paracontact metric manifold*. Notice that any such a semi-Riemannian metric is necessarily of signature $(n, n + 1)$ and the above condition (ii) of the definition of almost paracontact structures is automatically satisfied. Moreover, as in the almost contact case, from (2.13) it follows easily that $\eta = \tilde{g}(\cdot, \xi)$ and $\widetilde{g}(\cdot, \widetilde{\varphi}\cdot) = -\widetilde{g}(\widetilde{\varphi}\cdot, \cdot)$. Hence one defines the *fundamental 2-form* of the almost paracontact metric manifold by $\widetilde{\Phi}(X, Y) = \widetilde{g}(X, \widetilde{\varphi}Y)$. If $d\eta = \widetilde{\Phi}$, η becomes a contact form and $(M, \widetilde{\varphi}, \xi, \eta, \widetilde{g})$ is said to be a *paracontact metric manifold*.

On a paracontact metric manifold one defines the tensor field $\widetilde{h} := \frac{1}{2}\mathcal{L}_\xi\widetilde{\varphi}$. It was proved in [38] that $\widetilde{h}$ is a symmetric operator with respect to $\widetilde{g}$, it anti-commutes with $\widetilde{\varphi}$ and it vanishes identically if and only if ξ is a Killing vector field and in such case $(M, \widetilde{\varphi}, \xi, \eta, \widetilde{g})$ is called a *K-paracontact manifold*. Moreover the following identity holds:

$$(2.14) \qquad \widetilde{\nabla}\xi = -\widetilde{\varphi} + \widetilde{\varphi}\widetilde{h}.$$

A paracontact metric manifold is said to be *integrable*, or *para-CR*, if the following condition is satisfied, for any $X, Y \in \Gamma(TM)$,

$$(2.15) \qquad (\widetilde{\nabla}_X\widetilde{\varphi})Y = \eta(Y)(X - \widetilde{h}X) - \widetilde{g}(X - \widetilde{h}X, Y)\xi.$$

A normal paracontact metric manifold is said to be a *paraSasakian manifold*. Also in this context the paraSasakian condition implies the K-paracontact condition and the converse holds in dimension 3. In terms of the covariant derivative of $\widetilde{\varphi}$ the paraSasakian condition may be expressed by

$$(2.16) \qquad (\widetilde{\nabla}_X \widetilde{\varphi})Y = -\widetilde{g}(X,Y)\xi + \eta(Y)X.$$

Clearly, for K-paracontact manifolds the notion of integrability coincides with that of being paraSasakian. An equivalent definition of paraSasakian manifolds is presented in [3] in terms of pseudo-Riemannian cones. Standard examples of paraSasakian manifolds are the hyperboloid

$$\mathbb{H}^{2n+1}_{n+1}(1) = \left\{ (x_0, y_0, \ldots, x_n, y_n) \in \mathbb{R}^{2n+2} \mid x_0^2 + \ldots + x_n^2 - y_0^2 - \ldots - y_n^2 = 1 \right\}$$

and the hyperbolic Heisenberg group $\mathcal{H}^{2n+1} = \mathbb{R}^{2n} \times \mathbb{R}$ with the structures defined in [25]. Furthermore, let us recall that a notion of η-Einstein metric and $\mathcal{D}_c$-homothetic deformation can be introduced also in paracontact metric geometry ([38]). The definition is the same as the one given in (2.4) and (2.5) for contact Riemannian manifolds, with the only change that the constant of homothety c can be now any non-zero real number since the metric does not need to be positive definite.

3. The main results

There is a strict relationship between the theory of contact metric (κ, μ)-spaces and of paracontact geometry, as shown in [18] and [20]. In fact, given a non-Sasakian contact metric (κ, μ)-space $(M, \varphi, \xi, \eta, g)$, one can define in canonically two *integrable* paracontact metric structures on M, $(\widetilde{\varphi}_1, \xi, \eta, \widetilde{g}_1)$ and $(\widetilde{\varphi}_2, \xi, \eta, \widetilde{g}_2)$, which are compatible with the same underlying contact form and Reeb vector field as the (κ, μ)-space M. They are defined by

$$(3.1) \qquad \widetilde{\varphi}_1 := \frac{1}{\sqrt{1-\kappa}} \varphi h, \quad \widetilde{g}_1 := \frac{1}{\sqrt{1-\kappa}} g(\cdot, h\cdot) + \eta \otimes \eta,$$

$$(3.2) \qquad \widetilde{\varphi}_2 := \frac{1}{\sqrt{1-\kappa}} h, \quad \widetilde{g}_2 := \frac{1}{\sqrt{1-\kappa}} g(\cdot, \varphi h\cdot) + \eta \otimes \eta.$$

The curvature tensor fields of such paracontact metric structures, in turn, satisfy a nullity-like condition

$$\widetilde{R}_\alpha(X,Y)\xi = \widetilde{\kappa}_\alpha(\eta(Y)X - \eta(X)Y) + \widetilde{\mu}_\alpha(\eta(Y)\widetilde{h}_\alpha X - \eta(X)\widetilde{h}_\alpha Y),$$

where $\widetilde{\kappa}_1 = \left(1 - \frac{\mu}{2}\right) - 1$, $\widetilde{\mu}_1 = 2\left(1 - \sqrt{1-\kappa}\right)$ and $\widetilde{\kappa}_2 = \kappa - 2 + \left(1 - \frac{\mu}{2}\right)^2$, $\widetilde{\mu}_2 = 2$.

By (2.12) one can prove that

$$(3.3) \qquad \widetilde{h}_1 = -I_M h.$$

Hence, being integrable, the paracontact metric structure $(\widetilde{\varphi}_1, \xi, \eta, \widetilde{g}_1)$ is paraSasakian if and only if $I_M = 0$. Whereas for no value of κ and μ $(\widetilde{\varphi}_2, \xi, \eta, \widetilde{g}_2)$ is paraSasakian. Now we prove a much stronger result.

Theorem 3.1. *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian contact metric (κ, μ)-space such that $I_M \neq \pm 1$.*

(i) *If $|I_M| > 1$, then M admits a Sasakian structure, compatible with the contact form η, given by*

$$(3.4) \qquad \bar{\phi} := \epsilon \frac{1}{(1-\kappa)\sqrt{(2-\mu)^2 - 4(1-4\kappa)}} \mathcal{L}_\xi h \circ h, \qquad \bar{g} := -d\eta(\cdot, \bar{\phi}\cdot) + \eta \otimes \eta,$$

where

$$\epsilon := \begin{cases} 1 & \text{if } I_M > 1 \\ -1 & \text{if } I_M < -1 \end{cases}$$

(ii) *If $|I_M| < 1$, then M admits a paraSasakian structure, compatible with the contact form η, given by*

$$(3.5) \qquad \widetilde{\phi} := \frac{1}{(1-\kappa)\sqrt{4(1-\kappa)-(2-\mu)^2}}\mathcal{L}_\xi h \circ h, \quad \widetilde{g} := d\eta(\cdot, \widetilde{\phi}\cdot) + \eta \otimes \eta.$$

Proof. By formula (2.12), using (2.7) and the anti-commutativity of φ and h, one has that

$$(3.6) \qquad \mathcal{L}_\xi h \circ h = (1-\kappa)((2-\mu)\varphi + 2\varphi h).$$

Hence

$$\begin{aligned}
(\mathcal{L}_\xi h \circ h)^2 &= (2-\mu)^2(1-\kappa)^2\varphi^2 + 4(1-\kappa)^2\varphi h\varphi h + 2(2-\mu)(1-\kappa)^2\varphi\varphi h \\
&\quad + 2(2-\mu)(1-\kappa)^2\varphi h\varphi \\
&= (2-\mu)^2(1-\kappa)^2\varphi^2 - 4(1-\kappa)^2\varphi^2 h^2 \\
&= (1-\kappa)^2((2-\mu)^2 - 4(1-\kappa))\varphi^2.
\end{aligned}$$

Therefore $(\mathcal{L}_\xi h \circ h)^2 = \lambda^4\alpha(-I + \eta \otimes \xi)$, where we recall that $\lambda = \sqrt{1-\kappa}$ and we have put $\alpha := (2-\mu)^2 - 4(1-\kappa)$. Consequently, if $\alpha > 0$ then $\bar{\phi} := \epsilon\frac{1}{\lambda^2\sqrt{\alpha}}\mathcal{L}_\xi h \circ h$ defines an almost contact structure and if $\alpha < 0$ then $\widetilde{\phi} := \frac{1}{\lambda^2\sqrt{-\alpha}}\mathcal{L}_\xi h \circ h$ satisfies the first of the two conditions defining an almost paracontact structure. Notice that, since $\kappa \leq 1$, $\alpha > 0$ if and only if $|I_M| > 1$.

(i) Let us assume that $|I_M| > 1$. We firstly show that $\bar{\phi}$ is a normal almost contact structure. Just by using the definition of the tensor field N_ϕ, one can prove that for any almost contact structure (ϕ, ξ, η) (same contact form and Reeb vector field as (φ, ξ, η, g)) the following identity holds

$$(3.7) \qquad N_\phi(X, Y) = (\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X + (\nabla_X\phi)\phi Y - (\nabla_Y\phi)\phi X - \eta(Y)\nabla_X\xi + \eta(X)\nabla_Y\xi,$$

where ∇ is the Levi-Civita connection of the Riemannian metric g (notice that it is not required that the almost contact structure (ϕ, ξ, η) is compatible with g). By (3.6) we have that

$$(3.8) \qquad \bar{\phi} = \epsilon\frac{1}{\sqrt{\alpha}}((2-\mu)\varphi + 2\varphi h),$$

so that by (3.7) we find

$$\begin{aligned}
N_{\bar{\phi}}(X, Y) = {}& \frac{(2-\mu)^2}{\alpha}(\nabla_{\varphi X}\varphi)Y + \frac{2(2-\mu)}{\alpha}(\nabla_{\varphi h X}\varphi)Y + \frac{2(2-\mu)}{\alpha}(\nabla_{\varphi X}\varphi h)Y + \frac{4}{\alpha}(\nabla_{\varphi h X}\varphi h)Y \\
& - \frac{(2-\mu)^2}{\alpha}(\nabla_{\varphi Y}\varphi)X - \frac{2(2-\mu)}{\alpha}(\nabla_{\varphi h Y}\varphi)X - \frac{2(2-\mu)}{\alpha}(\nabla_{\varphi Y}\varphi h)X - \frac{4}{\alpha}(\nabla_{\varphi h Y}\varphi h)X \\
& + \frac{(2-\mu)^2}{\alpha}(\nabla_X\varphi)\varphi Y + \frac{2(2-\mu)}{\alpha}(\nabla_X\varphi)\varphi h Y + \frac{2(2-\mu)}{\alpha}(\nabla_X\varphi h)\varphi Y + \frac{4}{\alpha}(\nabla_X\varphi h)\varphi h Y \\
& - \frac{(2-\mu)^2}{\alpha}(\nabla_Y\varphi)\varphi X - \frac{2(2-\mu)}{\alpha}(\nabla_Y\varphi)\varphi h X - \frac{2(2-\mu)}{\alpha}(\nabla_Y\varphi h)\varphi X - \frac{4}{\alpha}(\nabla_Y\varphi h)\varphi h X \\
(3.9) \qquad & - \eta(Y)\nabla_X\xi + \eta(X)\nabla_Y\xi.
\end{aligned}$$

Then taking (2.3), (2.9) and (2.10) into account, and using (2.7) and the anti-commutativity of φ and h, after very long computations one can prove that (3.9) reduces to

$$\begin{aligned}
N_{\bar{\phi}}(X, Y) = {}& \left(1 - \frac{(2-\mu)^2}{\alpha} + \frac{4\lambda^2(2-\mu)}{\alpha} - \frac{4\lambda^2}{\alpha} + \frac{4\lambda^2\mu}{\alpha}\right)(\eta(Y)\varphi X - \eta(X)\varphi Y) \\
(3.10) \qquad & + \left(1 + \frac{(2-\mu)^2}{\alpha} - \frac{4(2-\mu)}{\alpha} + \frac{4\lambda^2}{\alpha} + \frac{2(2-\mu)\mu}{\alpha}\right)(\eta(Y)\varphi h X - \eta(X)\varphi h Y).
\end{aligned}$$

By substituting the values of λ and α in (3.10) one can check that in fact $N_{\bar\phi}(X,Y) = 0$. Next we prove that the tensor $\bar g$ in (3.4) defines a Riemannian metric compatible with the almost contact structure $(\bar\phi, \xi, \eta)$ and such that $d\eta = \bar g(\cdot, \bar\varphi\cdot)$. Firstly notice that from (3.4) and (3.6) it follows that

$$\bar g(X,Y) = \epsilon \frac{1}{\sqrt{(2-\mu)^2 - 4(1-\kappa)}} \left((2-\mu)g(X,Y) + 2g(X,hY)\right)$$

(3.11)
$$+ \left(1 - \epsilon \frac{2-\mu}{\sqrt{(2-\mu)^2 - 4(1-\kappa)}}\right) \eta(X)\eta(Y),$$

where

$$\epsilon := \begin{cases} 1 & \text{if } I_M > 1 \\ -1 & \text{if } I_M < -1 \end{cases}$$

Since h is a symmetric operator with respect to the Riemannian metric g, by (3.11) we see immediately that $\bar g$ is a symmetric tensor. Next we prove that it is positive definite. First, by the very definition of $\bar g$, one has that $\bar g(\xi, \xi) = 1$. Moreover, for any non-zero tangent vector field $X \in \Gamma(\mathcal D)$, one has by (3.6) that

$$\bar g(X,X) = \epsilon \frac{1}{\sqrt{(2-\mu)^2 - 4(1-\kappa)}} \left((2-\mu)g(X,X) + 2g(X,hX)\right).$$

We can distinguish two cases: $X \in \Gamma(\mathcal D_h(\lambda))$ or $X \in \Gamma(\mathcal D_h(-\lambda))$. In the first case we obtain

$$\bar g(X,X) = \epsilon \frac{(2-\mu) + 2\sqrt{1-\kappa}}{\sqrt{(2-\mu)^2 - 4(1-\kappa)}} g(X,X).$$

The above formula implies that $\bar g(X,X) > 0$, since $(2-\mu) + 2\sqrt{1-\kappa}$ is always positive when $I_M > 1$ and negative when $I_M < -1$, the same as ϵ. The other case $X \in \Gamma(\mathcal D_h(-\lambda))$ is similar. It remains to be seen that $\bar g$ is an *associated metric*, that is $d\eta(X,Y) = \bar g(X,\bar\phi Y)$ for all $X,Y \in \Gamma(TM)$. Indeed by the definition of $\bar g$ and by the property of $\bar\phi$ of defining an almost contact structure we have $\bar g(X,\bar\phi Y) = -d\eta(X,\bar\phi^2 Y) + \eta(X)\eta(\bar\phi Y) = -d\eta(X,-Y+\eta(Y)\xi) = d\eta(X,Y) + \eta(Y)d\eta(X,\xi) = d\eta(X,Y)$.
(ii) Now let us assume that $|I_M| < 1$. In this case $\widetilde\phi$ is given by

(3.12)
$$\widetilde\phi = \frac{1}{\sqrt{-\alpha}}\left((2-\mu)\varphi + 2\varphi h\right)$$

and the tensor $\widetilde g$ by

(3.13)
$$\widetilde g(X,Y) = -\frac{1}{\sqrt{-\alpha}}\left((2-\mu)g(X,Y) + 2g(X,hY)\right) + \left(1 + \frac{2-\mu}{\sqrt{-\alpha}}\right)\eta(X)\eta(Y).$$

Equation (3.13) easily implies that $\widetilde g$ is a symmetric tensor. Moreover, one can prove as in the case (i) that $d\eta(X,Y) = \widetilde g(X,\widetilde\phi Y)$ for any $X,Y \in \Gamma(TM)$. This last condition, together with $\widetilde\phi^2 = I - \eta \otimes \xi$, implies that $\widetilde g(\widetilde\phi X, \widetilde\phi Y) = -\widetilde g(X,Y) + \eta(X)\eta(Y)$. Thus $\widetilde g$ is a semi-Riemannian metric of signature $(n, n+1)$ and $(\widetilde\phi, \xi, \eta, \widetilde g)$ is a paracontact metric manifold. So, once we have proved that the structure $(\widetilde\phi, \xi, \eta)$ is normal, we would have proved that $(M, \widetilde\phi, \xi, \eta, \widetilde g)$ is a paraSasakian manifold. In order to do that, we will check that the almost paracontact structure $(\widetilde\phi, \xi, \eta)$ satisfies the conditions of Theorem 2.2. First we prove that the eigendistributions $\mathcal D^\pm$ are integrable. Let $X, X' \in \Gamma(\mathcal D^+)$, i.e. $\widetilde\phi X = X$, $\widetilde\phi X' = X'$

Then we have

$$\widetilde{\phi}[X, X'] = \frac{2-\mu}{\sqrt{-\alpha}}\varphi[X, X'] + \frac{2}{\sqrt{-\alpha}}\varphi h[X, X']$$

$$= \frac{2-\mu}{\sqrt{-\alpha}}\varphi\nabla_X X' - \frac{2-\mu}{\sqrt{-\alpha}}\varphi\nabla_{X'}X + \frac{2}{\sqrt{-\alpha}}\varphi h\nabla_X X' - \frac{2}{\sqrt{-\alpha}}\varphi h\nabla_{X'}X$$

$$= \frac{2-\mu}{\sqrt{-\alpha}}\nabla_X\varphi X' - \frac{2-\mu}{\sqrt{-\alpha}}(\nabla_X\varphi)X' - \frac{2-\mu}{\sqrt{-\alpha}}\nabla_{X'}\varphi X + \frac{2-\mu}{\sqrt{-\alpha}}(\nabla_{X'}\varphi)X$$

$$+ \frac{2}{\sqrt{-\alpha}}\nabla_X\varphi h X' - \frac{2}{\sqrt{-\alpha}}(\nabla_X\varphi h)X' - \frac{2}{\sqrt{-\alpha}}\nabla_{X'}\varphi h X + \frac{2}{\sqrt{-\alpha}}(\nabla_{X'}\varphi h)X$$

$$= \nabla_X\left(\frac{2-\mu}{\sqrt{-\alpha}}\varphi X' + \frac{2}{\sqrt{-\alpha}}\varphi h X'\right) - \nabla_{X'}\left(\frac{2-\mu}{\sqrt{-\alpha}}\varphi X + \frac{2}{\sqrt{-\alpha}}\varphi h X\right)$$

$$- \frac{2-\mu}{\sqrt{-\alpha}}(\nabla_X\varphi)X' + \frac{2-\mu}{\sqrt{-\alpha}}(\nabla_{X'}\varphi)X - \frac{2}{\sqrt{-\alpha}}(\nabla_X\varphi h)X' + \frac{2}{\sqrt{-\alpha}}(\nabla_{X'}\varphi h)X.$$

Thus by using (2.8) and (2.10) we get

$$\widetilde{\phi}[X, X'] = \nabla_X\widetilde{\phi}X' - \nabla_{X'}\widetilde{\phi}X - \frac{2-\mu}{\sqrt{-\alpha}}g(X + hX, X')\xi + \frac{2-\mu}{\sqrt{-\alpha}}g(X' + hX', X)\xi$$

$$- \frac{2}{\sqrt{-\alpha}}\left(g(X, hX')\xi + (1-\kappa)g(X, X')\xi\right) + \frac{2}{\sqrt{-\alpha}}\left(g(X', hX)\xi + (1-\kappa)g(X', X)\xi\right)$$

$$= \nabla_X X' - \nabla_{X'}X$$

$$= [X, X'].$$

Hence $[X, X'] \in \Gamma(\mathcal{D}^+)$ and we conclude that $\mathcal{D}^+$ is an integrable distribution. It remains to be seen that ξ is a foliated vector field with respect to the foliation defined by $\mathcal{D}^+$, i.e. $[\xi, X] \in \Gamma(\mathcal{D}^+)$ for any $X \in \Gamma(\mathcal{D}^+)$. By using (2.3), $\nabla_\xi\varphi = 0$ and $\nabla_\xi\varphi h = \mu h$ we have

$$\widetilde{\phi}[\xi, X] = \widetilde{\phi}\nabla_\xi X - \widetilde{\phi}\nabla_X\xi$$

$$= \frac{2-\mu}{\sqrt{-\alpha}}\varphi\nabla_\xi X + \frac{2}{\sqrt{-\alpha}}\varphi h\nabla_\xi X + \widetilde{\phi}\varphi X + \widetilde{\phi}\varphi h X$$

$$= \frac{2-\mu}{\sqrt{-\alpha}}\nabla_\xi\varphi X + \frac{2}{\sqrt{-\alpha}}\nabla_\xi\varphi h X - \frac{2\mu}{\sqrt{-\alpha}}hX + \frac{2-\mu}{\sqrt{-\alpha}}\varphi^2 X + \frac{2}{\sqrt{-\alpha}}\varphi h\varphi X$$

$$+ \frac{2-\mu}{\sqrt{-\alpha}}\varphi^2 h + \frac{2}{\sqrt{-\alpha}}\varphi h\varphi h X$$

(3.14) $$= \nabla_\xi\left(\frac{2-\mu}{\sqrt{-\alpha}}\varphi X + \frac{2}{\sqrt{-\alpha}}\varphi h X\right) - \frac{\mu}{\sqrt{-\alpha}}hX + \frac{\mu - 2k}{\sqrt{-\alpha}}X.$$

On the other hand, $\frac{2-\mu}{\sqrt{-\alpha}}\varphi X + \frac{2}{\sqrt{-\alpha}}\varphi h X = \widetilde{\phi}X$, and one can check that $\nabla_{\widetilde{\phi}X}\xi = -\varphi\widetilde{\phi}X - \varphi h\widetilde{\phi}X = \frac{\mu}{\sqrt{-\alpha}}hX - \frac{\mu-2k}{\sqrt{-\alpha}}X$. So that (3.14) becomes

$$\widetilde{\phi}[\xi, X] = \nabla_\xi\widetilde{\phi}X - \nabla_{\widetilde{\phi}X}\xi = \nabla_\xi X - \nabla_X\xi = [\xi, X],$$

that is $[\xi, X] \in \Gamma(\mathcal{D}^+)$. The same also arguments work for $\mathcal{D}^-$. Therefore, by Theorem 2.2 we conclude that the paracontact metric structure $(\widetilde{\phi}, \xi, \eta, g)$ is normal and hence paraSasakian. $\square$

Remark 3.2. One can see that the metrics stated in Theorem 3.1 extends the Riemannian metrics introduced, in a different context, in [18] and [20].

Remark 3.3. It is interesting to notice that the metrics $\bar{g}$ and $\widetilde{g}$ are invariant under $\mathcal{D}$-homothetic deformations. Indeed let us apply a $\mathcal{D}_c$-homothetic deformation (2.5) to the non-Sasakian contact metric (κ, μ)-space $(M, \varphi, \xi, \eta, g)$, for some $c > 0$. Then we obtain a new contact metric (κ', μ')-structure $(\varphi', \xi', \eta', g')$ on M, with κ' and μ' given by (2.11) and the same Boeckx invariant as (φ, ξ, η, g). Thus $(\varphi', \xi', \eta', g')$ in turn satisfies the assumptions of Theorem 3.1 and admits a Sasakian structure $(\bar{\phi}', \xi', \eta', \bar{g}')$ or a paraSasakian structure $(\widetilde{\phi}', \xi', \eta', \widetilde{g}')$ according to the circumstance that $|I_M| > 1$ or $|I_M| < 1$, respectively. Let us assume for instance that $|I_M| > 1$. Since $h' = \frac{1}{c}h$ and a straightforward computation shows that $\alpha' = (2 - \mu')^2 - 4(1 - \kappa') = \frac{1}{c^2}\alpha$, from (3.8) it follows that

$$\bar{\phi}' = \epsilon \frac{1}{\sqrt{\alpha'}}\left((2 - \mu')\varphi' + 2\varphi'h'\right) = \epsilon\frac{1}{\sqrt{\alpha}}\left((2 - \mu)\varphi + 2\varphi h\right) = \bar{\phi}.$$

Moreover, due to $\bar{g}'(\cdot, \bar{\phi}'\cdot) = d\eta'$, one easily proves that $\bar{g}' = c\bar{g} + c(c-1)\eta \otimes \eta$. Thus the Sasakian structure $(\bar{\phi}', \xi', \eta', \bar{g}')$, associated via Theorem 3.1 to the (κ, μ)-space $(\varphi', \xi', \eta', g')$, is nothing but the structure $\mathcal{D}_c$-homothetic to the Sasakian structure $(\bar{\phi}, \xi, \eta, \bar{g})$. The same arguments work in the case $|I_M| < 1$.

We will now calculate the curvature tensor of the Sasakian and paraSasakian manifolds that appear in the previous theorem, which we will show to be always η-Einstein.

Theorem 3.4. *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian contact metric (κ, μ)-space such that $I_M \neq \pm 1$.*

(i) *If $|I_M| > 1$ then the Levi-Civita connection of the Sasakian structure $(M, \bar{\phi}, \xi, \eta, \bar{g})$ defined in (3.4) relates to the original one the following way:*

(3.15)
$$\begin{aligned}\overline{\nabla}_X Y =&\nabla_X Y + \left(1 - \epsilon\frac{2 - \mu}{\sqrt{\alpha}}\right)(\eta(Y)\varphi X + \eta(X)\varphi Y) \\ &+ \left(1 - \epsilon\frac{2}{\sqrt{\alpha}}\right)(\eta(Y)\varphi hX + \eta(X)\varphi hY) - g(X, \varphi hY)\xi,\end{aligned}$$

where, as in the previous theorem,

$$\epsilon := \left\{\begin{array}{ll} 1 & if \ I_M > 1 \\ -1 & if \ I_M < -1\end{array}\right.$$

Moreover, its curvature tensor has the form

(3.16)
$$\begin{aligned}\overline{R}(X, Y)Z =& \ \epsilon\frac{\sqrt{\alpha}}{2}(\bar{g}(Y, Z)X - \bar{g}(X, Z)Y) \\ &+ \left(\epsilon\frac{\sqrt{\alpha}}{2} - 1\right)(\bar{g}(X, \bar{\phi}Z)\bar{\phi}Y - \bar{g}(Y, \bar{\phi}Z)\bar{\phi}X + 2\bar{g}(X, \bar{\phi}Y)\bar{\phi}Z) \\ &+ \left(\epsilon\frac{\sqrt{\alpha}}{2} - 1\right)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \bar{g}(X, Z)\eta(Y)\xi - \bar{g}(Y, Z)\eta(X)\xi) \\ &+ \epsilon\frac{\sqrt{\alpha}}{2(1 - \kappa)}\left(\bar{g}(hY, Z)hX - \bar{g}(hX, Z)hY - \bar{g}(hY, \bar{\phi}Z)\bar{\phi}hX + \bar{g}(hX, \bar{\phi}Z)\bar{\phi}hY\right).\end{aligned}$$

The Ricci tensor of M^{2n+1} is

(3.17)
$$\overline{\mathrm{Ric}} = (\epsilon n\sqrt{\alpha} - 2)\bar{g} + (-\epsilon n\sqrt{\alpha} + 2n + 2)\eta \otimes \eta$$

and therefore the Sasakian structure is always η-Einstein.

(ii) *If $|I_M| < 1$ then the paraSasakian structure $(M, \widetilde{\phi}, \xi, \eta, \bar{g})$ defined in (3.5) has the following Levi-Civita connection:*

(3.18)
$$\widetilde{\nabla}_X Y = \nabla_X Y + \left(1 - \frac{2 - \mu}{\sqrt{-\alpha}}\right)(\eta(Y)\varphi X + \eta(X)\varphi Y)$$
$$+ \left(1 - \frac{2}{\sqrt{-\alpha}}\right)(\eta(Y)\varphi h X + \eta(X)\varphi h Y) - g(X, \varphi h Y)\xi.$$

Furthermore, the curvature tensor can be written as

(3.19)
$$\widetilde{R}(X, Y)Z = -\frac{\sqrt{-\alpha}}{2}(\widetilde{g}(Y, Z)X - \widetilde{g}(X, Z)Y)$$
$$+ \left(\frac{\sqrt{-\alpha}}{2} - 1\right)(\widetilde{g}(X, \widetilde{\phi}Z)\widetilde{\phi}Y - \widetilde{g}(Y, \widetilde{\phi}Z)\widetilde{\phi}X + 2\widetilde{g}(X, \widetilde{\phi}Y)\widetilde{\phi}Z)$$
$$- \left(\frac{\sqrt{-\alpha}}{2} - 1\right)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \widetilde{g}(X, Z)\eta(Y)\xi - \widetilde{g}(Y, Z)\eta(X)\xi)$$
$$- \frac{\sqrt{-\alpha}}{2(1 - \kappa)}\left(\widetilde{g}(hY, Z)hX - \widetilde{g}(hX, Z)hY + \widetilde{g}(hY, \widetilde{\phi}Z)\widetilde{\phi}hX - \widetilde{g}(hX, \widetilde{\phi}Z)\widetilde{\phi}hY\right).$$

The Ricci tensor of M^{2n+1} is

(3.20)
$$\widetilde{\mathrm{Ric}} = (-n\sqrt{-\alpha} + 3)\widetilde{g} + (n\sqrt{-\alpha} - 2n - 3)\eta \otimes \eta$$

and therefore the paraSasakian structure is always η-Einstein.

Proof. **(i)** Bearing in mind that $\overline{\nabla}$ is a Levi-Civita connection, we can use Koszul's formula and deduce

$$2\bar{g}(\overline{\nabla}_X Y, Z) = X\bar{g}(Y, Z) + Y\bar{g}(X, Z) - Z\bar{g}(X, Y) + \bar{g}([X, Y]Z) + \bar{g}([Z, X], Y) - \bar{g}([Y, Z], X) =$$

$$= \epsilon \frac{2(2 - \mu)}{\sqrt{\alpha}} g(\nabla_X Y, Z)$$

$$+ 2\left(1 - \epsilon \frac{2 - \mu}{\sqrt{\alpha}}\right)(\eta(Z)\eta(\nabla_X Y) + \eta(Y)g(X, \varphi Z) + \eta(X)g(Y, \varphi Z) - \eta(Z)g(X, \varphi h Y))$$

$$+ \frac{2\epsilon}{\sqrt{\alpha}}(Xg(Y, hZ) + Yg(X, hZ) - Zg(X, hY) + g([X, Y], hZ) + g([Z, X], Y) - g([Y, Z], hX)).$$

As $\nabla g = 0$ and equation (2.9) holds, we obtain that

$$Xg(Y, hZ) + Yg(X, hZ) - Zg(X, hY) + g([X, Y], hZ) + g([Z, X], Y) - g([Y, Z], hX) =$$
$$= 2g(\nabla_X Y, hZ) + 2(1 - \kappa)(\eta(X)g(Y, \varphi X) + \eta(Y)g(X, \varphi Z))$$
$$+ (\mu - 2)\eta(Z)g(X, \varphi h Y) - \mu(\eta(X)g(Y, \varphi h Z) + \eta(Y)g(X, \varphi h Z)),$$

which substituting in the previous equation, and using (3.11), gives us that

(3.21)
$$2\bar{g}(\overline{\nabla}_X Y, Z) = 2\bar{g}(\nabla_X Y, Z) + 2\left(1 + \epsilon \frac{\mu - 2\kappa}{\sqrt{\alpha}}\right)(\eta(Y)g(X, \varphi Z) + \eta(X)g(Y, \phi Z))$$
$$- 2\eta(Z)g(X, \phi h Y) - \epsilon \frac{2\mu}{\sqrt{\alpha}}(\eta(X)g(Y, \phi h Z) + \eta(Y)g(X, \phi h Z)).$$

We recall now that η is the contact form of both structures, so $g(X, \varphi Y) = d\eta(X, Y) = \bar{g}(X, \bar{\phi}Y)$ and $g(Y, \phi h Z) = -\bar{g}(hY, \bar{\phi}Z)$ is deduced. Substituting both formulas in (3.21), it follows that

$$\overline{\nabla}_X Y = \nabla_X Y + \left(1 + \epsilon \frac{\mu - 2\kappa}{\sqrt{\alpha}}\right)(\eta(Y)\bar{\phi}X + \eta(X)\bar{\phi}Y)$$
$$- g(X, \phi h Y)\xi - \epsilon \frac{\mu}{\sqrt{\alpha}}(\eta(X)\bar{\phi}h Y + \eta(Y)\bar{\phi}h X).$$

This last equation together with (3.8) gives us (3.15).

As $\overline{R}$ is a Riemannian curvature tensor, using equations (2.8) and (2.10) we obtain after long computations that

$$\overline{R}(X,Y)Z = R(X,Y)Z + \left(1 - \epsilon\frac{2-\mu}{\sqrt{\alpha}}\right)(g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z)$$

$$+ g(Y,\varphi hZ)\varphi X - g(X,\varphi hZ)\varphi Y$$

$$+ \left(1 - \epsilon\frac{2}{\sqrt{\alpha}}\right)(g(\varphi Y,Z)\varphi hX - g(\varphi X,Z)\varphi hZ + 2g(X,\varphi Y)\varphi hZ)$$

$$+ (1-\kappa)(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) + \left(\kappa - \epsilon\frac{2-\mu}{\sqrt{\alpha}}\right)(\eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi)$$

(3.22)
$$- \mu(\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY) + \left(\mu - \epsilon\frac{2}{\sqrt{\alpha}}\right)(\eta(Y)g(X,hZ)\xi - \eta(X)g(Y,hZ)\xi).$$

On the other hand, we know from [9] that the curvature tensor of a non-Sasakian contact metric (κ,μ)-space is

$$R(X,Y)Z = \left(1 - \frac{\mu}{2}\right)(g(Y,X)X - g(X,Z)Y)$$

$$- \frac{\mu}{2}(g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z)$$

$$+ \left(1 - \frac{\mu}{2} - \kappa\right)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi)$$

$$+ g(Y,X)hX - g(X,Z)hY + g(hY,X)X - g(hX,Z)Y$$

$$+ \frac{1-\frac{\mu}{2}}{1-\kappa}(g(hY,X)hX - g(hX,Z)hY) + \frac{\kappa-\frac{\mu}{2}}{1-\kappa}(g(\varphi hY,X)\varphi hX - g(\varphi hX,Z)\varphi hY)$$

(3.23)
$$+ (1-\mu)(\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi).$$

Therefore, substituting in (3.22) gives us

$$\overline{R}(X,Y)Z = \left(1 - \frac{\mu}{2}\right)(g(Y,X)X - g(X,Z)Y)$$

$$+ \left(1 - \epsilon\frac{2-\mu}{\sqrt{\alpha}} - \frac{\mu}{2}\right)(g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z)$$

$$- \frac{\mu}{2}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) + \left(1 - \epsilon\frac{2-\mu}{\sqrt{\alpha}} - \frac{\mu}{2}\right)(g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi)$$

$$+ g(Y,X)hX - g(X,Z)hY + g(hY,X)X - g(hX,Z)Y$$

$$+ \frac{1-\frac{\mu}{2}}{1-\kappa}(g(hY,X)hX - g(hX,Z)hY + g(\varphi hY,X)\varphi hX - g(\varphi hX,Z)\varphi hY)$$

$$+ \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + \left(1 - \epsilon\frac{2}{\sqrt{\alpha}}\right)(g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi)$$

$$+ \left(1 - \epsilon\frac{2}{\sqrt{\alpha}}\right)(g(\varphi Y,Z)\varphi hX - g(\varphi X,Z)\varphi hZ + 2g(X,\varphi Y)\varphi hZ)$$

$$+ g(Y,\varphi hZ)\varphi X - g(X,\varphi hZ)\varphi Y$$

Thus by using (3.8) and (3.11) we get (3.16).

We will finally compute the Ricci tensor, for which we will construct an orthonormal basis with respect to the structure $(M,\bar{\phi},\xi,\eta,\bar{g})$. Let us take a φ-basis $\{X_i, Y_i = \varphi X_i, \xi\}$, $i = 1,\dots,n$, such that $X_i \in \mathcal{D}_h(\lambda)$ (and therefore $Y_i \in \mathcal{D}_h(-\lambda)$), which always exists because (M,ϕ,ξ,η,g) is a non-Sasakian contact metric

(κ,μ)-space. Then we define

$$(3.24) \qquad \overline{X}_i = \frac{1}{\sqrt{\gamma}}X_i, \quad \overline{Y}_i = \sqrt{\gamma}Y_i,$$

where $\gamma = \epsilon\frac{2-\mu+2\lambda}{\sqrt{\alpha}}$. Notice that $\gamma > 0$ because by assumption $|I_M| > 1$ and that we can write $\frac{1}{\gamma} = \epsilon\frac{2-\mu-2\lambda}{\sqrt{\alpha}}$. Thus $\{\overline{X}_i, \overline{Y}_i, \xi\}$ is a $\bar{\phi}$-basis with $\overline{X}_i \in \mathcal{D}_h(\lambda)$ and $\overline{Y}_i \in \mathcal{D}_h(-\lambda)$. By the definition of the Ricci tensor, we have that

$$(3.25) \qquad \overline{\mathrm{Ric}}(X,Y) = \sum_{i=1}^{n}(\bar{g}(\overline{R}(X,\overline{X}_i)\overline{X}_i,Y) + \bar{g}(\overline{R}(X,\overline{Y}_i)\overline{Y}_i,Y)) + \bar{g}(\overline{R}(X,\xi)\xi,Y).$$

Using formula (3.16), we can now compute

$$\bar{g}(\overline{R}(X,\overline{X}_i)\overline{X}_i,Y) = \epsilon\frac{\sqrt{\alpha}}{2}\bar{g}(X,Y) - \left(\epsilon\frac{\sqrt{\alpha}}{2} - 1\right)\eta(X)\eta(Y) + \epsilon\frac{\sqrt{\alpha}}{2\lambda}\bar{g}(hX,Y)$$
$$- \epsilon\sqrt{\alpha}\bar{g}(X,\overline{X}_i)\bar{g}(\overline{X}_i,Y) + (\epsilon\sqrt{\alpha} - 3)\bar{g}(X,\overline{Y}_i)\bar{g}(\overline{Y}_i,Y),$$
$$\bar{g}(\overline{R}(X,\overline{Y}_i)\overline{Y}_i,Y) = \epsilon\frac{\sqrt{\alpha}}{2}\bar{g}(X,Y) - \left(\epsilon\frac{\sqrt{\alpha}}{2} - 1\right)\eta(X)\eta(Y) - \epsilon\frac{\sqrt{\alpha}}{2\lambda}\bar{g}(hX,Y)$$
$$+ (\epsilon\sqrt{\alpha} - 3)\bar{g}(X,\overline{X}_i)\bar{g}(\overline{X}_i,Y) - \epsilon\sqrt{\alpha}\bar{g}(X,\overline{Y}_i)\bar{g}(\overline{Y}_i,Y),$$
$$\bar{g}(\overline{R}(X,\xi)\xi,Y) = \bar{g}(X,Y) - \eta(X)\eta(Y),$$

which substituting in (3.25) gives us (3.17).

(ii) As in the previous case, we use Koszul's formula, $\nabla g = 0$ and the equation (2.9), which gives us

$$2\tilde{g}(\widetilde{\nabla}_X Y, Z) = X\tilde{g}(Y,Z) + Y\tilde{g}(X,Z) - Z\tilde{g}(X,Y) + \tilde{g}([X,Y]Z) + \tilde{g}([Z,X],Y) - \tilde{g}([Y,Z],X)$$
$$= 2\tilde{g}(\nabla_X Y, Z) + 2\left(1 + \frac{2\kappa - \mu}{\sqrt{-\alpha}}\right)(\eta(Y)g(X,\varphi Z) + \eta(X)g(Y,\varphi Z))$$
$$- 2\eta(Z)g(X,\varphi hY) + \frac{2\mu}{\sqrt{-\alpha}}(\eta(X)g(Y,\varphi hZ) + \eta(Y)g(X,\varphi hZ)).$$

As η is also the contact form of the paraSasakian structure, we have that $g(X,\varphi Y) = d\eta(X,Y) = \tilde{g}(X,\tilde{\phi}Y)$ and $g(Y,\phi hZ) = -\tilde{g}(hY,\tilde{\phi}Z)$. Substituting both formulas in the previous one and using that $\tilde{g}$ is a non-degenerate metric, it follows that

$$\widetilde{\nabla}_X Y = \nabla_X Y - \left(1 + \frac{\mu - 2\kappa}{\sqrt{-\alpha}}\right)(\eta(Y)\tilde{\phi}X + \eta(X)\tilde{\phi}Y)$$
$$- g(X,\phi hY)\xi + \frac{\mu}{\sqrt{-\alpha}}(\eta(X)\tilde{\phi}hY + \eta(Y)\tilde{\phi}hX).$$

This last equation together with (3.12) gives us (3.18).

Equations (3.15) and (3.18) only vary in the sign of α, hence the first computations barely change (except in the sign of α) and we get after some more computations that

$$
\begin{aligned}
\widetilde{R}(X,Y)Z = {}& R(X,Y)Z + \left(1 - \frac{2-\mu}{\sqrt{-\alpha}}\right)(g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z) \\
& + g(Y,\varphi hZ)\varphi X - g(X,\varphi hZ)\varphi Y \\
& + \left(1 - \frac{2}{\sqrt{-\alpha}}\right)(g(\varphi Y,Z)\varphi hX - g(\varphi X,Z)\varphi hZ + 2g(X,\varphi Y)\varphi hZ) \\
& - (1+\kappa)(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) + \left(\kappa - \frac{2-\mu}{\sqrt{-\alpha}}\right)(\eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi) \\
& - \mu(\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY) + \left(\mu - \frac{2}{\sqrt{-\alpha}}\right)(\eta(Y)g(X,hZ)\xi - \eta(X)g(Y,hZ)\xi),
\end{aligned}
$$

(3.26)

which bears a remarkable similarity to (3.22). Substituting formula (3.23) in (3.26) gives us

$$
\begin{aligned}
\widetilde{R}(X,Y)Z = {}& \left(1 - \frac{\mu}{2}\right)(g(Y,X)X - g(X,Z)Y) \\
& + \left(1 - \frac{2-\mu}{\sqrt{-\alpha}} - \frac{\mu}{2}\right)(g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z) \\
& \left(2 - \frac{\mu}{2}\right)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) + \left(1 - \frac{2-\mu}{\sqrt{-\alpha}} - \frac{\mu}{2}\right)(g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi) \\
& + g(Y,X)hX - g(X,Z)hY + g(hY,X)X - g(hX,Z)Y \\
& + \frac{1 - \frac{\mu}{2}}{1 - \kappa}(g(hY,X)hX - g(hX,Z)hY + g(\varphi hY,X)\varphi hX - g(\varphi hX,Z)\varphi hY) \\
& + \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + \left(1 - \frac{2}{\sqrt{-\alpha}}\right)(g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi) \\
& + \left(1 - \frac{2}{\sqrt{-\alpha}}\right)(g(\varphi Y,Z)\varphi hX - g(\varphi X,Z)\varphi hZ + 2g(X,\varphi Y)\varphi hZ) \\
& + g(Y,\varphi hZ)\varphi X - g(X,\varphi hZ)\varphi Y.
\end{aligned}
$$

Hence (3.19) is obtained after using (3.12) and (3.13).

It remains to be proved the formula of the Ricci tensor. As in the previous case, we well first have to construct an orthonormal basis. We take again a φ-basis $\{X_i, Y_i = \varphi X_i, \xi\}$, $i = 1, \dots, n$, such that $X_i \in \mathcal{D}_h(\lambda)$ and $Y_i \in \mathcal{D}_h(-\lambda)$. We then define

$$
\widetilde{X}_i = \frac{1}{\sqrt{\gamma}}X_i, \quad \widetilde{Y}_i = \sqrt{\gamma}Y_i,
$$

where $\gamma = \frac{2-\mu+2\lambda}{\sqrt{-\alpha}}$. We know that $\gamma > 0$ because $|I_M| < 1$ and that $\frac{1}{\gamma} = -\frac{2-\mu-2\lambda}{\sqrt{-\alpha}}$. Therefore, $\{\widetilde{X}_i, \widetilde{Y}_i, \xi\}$ is a $\widetilde{\phi}$-basis with $\widetilde{X}_i \in \mathcal{D}_h(\lambda)$ and $\widetilde{Y}_i \in \mathcal{D}_h(-\lambda)$. Let us notice that $\widetilde{g}(\widetilde{X}_i, \widetilde{X}_i) = -1$ and that $\widetilde{g}(\widetilde{X}_i, \widetilde{X}_i) = 1$ for all $i \in \{1, \dots, n\}$. Thus by the definition of the Ricci tensor

(3.27) $$\widetilde{\mathrm{Ric}}(X,Y) = \sum_{i=1}^{n}(-\widetilde{g}(\widetilde{R}(X,\widetilde{X}_i)\widetilde{X}_i, Y) + \widetilde{g}(\widetilde{R}(X,\widetilde{Y}_i)\widetilde{Y}_i, Y)) + \widetilde{g}(\widetilde{R}(X,\xi)\xi, Y).$$

Using now equation (3.19), we have that

$$\widetilde{g}(\widetilde{R}(X, \widetilde{X}_i)\widetilde{X}_i, Y) = \frac{\sqrt{-\alpha}}{2}\widetilde{g}(X, Y) - \left(\frac{\sqrt{-\alpha}}{2} - 1\right)\eta(X)\eta(Y) + \frac{\sqrt{-\alpha}}{2\lambda}\widetilde{g}(hX, Y)$$
$$+ \sqrt{-\alpha}\widetilde{g}(X, \widetilde{X}_i)\widetilde{g}(\widetilde{X}_i, Y) + (\sqrt{-\alpha} - 3)\widetilde{g}(X, \widetilde{Y}_i)\widetilde{g}(\widetilde{Y}_i, Y),$$

$$\widetilde{g}(\widetilde{R}(X, \widetilde{Y}_i)\widetilde{Y}_i, Y) = -\frac{\sqrt{-\alpha}}{2}\widetilde{g}(X, Y) + \left(\frac{\sqrt{-\alpha}}{2} - 1\right)\eta(X)\eta(Y) + \frac{\sqrt{-\alpha}}{2\lambda}\widetilde{g}(hX, Y)$$
$$+ (\sqrt{-\alpha} - 3)\widetilde{g}(X, \widetilde{X}_i)\widetilde{g}(\widetilde{X}_i, Y) + \sqrt{-\alpha}\widetilde{g}(X, \widetilde{Y}_i)\widetilde{g}(\widetilde{Y}_i, Y),$$

$$\widetilde{g}(\widetilde{R}(X, \xi)\xi, Y) = -\widetilde{g}(X, Y) + \eta(X)\eta(Y),$$

which substituting in (3.27) gives us (3.20). □

Remark 3.5. It is worth noticing that the first three tensors that appear in equations (3.16) and (3.19) are well-known because a Sasakian space form (M, ϕ, ξ, η, g) of constant ϕ-sectional curvature has the following curvature tensor (cf. [7]):

$$R(X, Y)Z = \frac{c+3}{4}(g(Y, X)X - g(X, Z)Y)$$
$$+ \frac{c-1}{4}(g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z)$$
$$+ \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi)$$

These tensors are sometimes denoted (cf. [2]) as R_1, R_2 and R_3 and the writing of the curvature tensor simplified to

$$R = \frac{c+3}{4}R_1 + \frac{c-1}{4}R_2 + \frac{c-1}{4}R_3.$$

By the previous theorem, we can now compute the sectional curvature:

Corollary 3.6. *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian contact metric (κ, μ)-space such that $I_M \neq \pm 1$.*

(i) *If $|I_M| > 1$, the sectional curvature of the Sasakian structure that M admits, defined as in (3.4), is:*

(3.28)
$$\overline{K}(X, Y) = \begin{cases} 1 & \text{if } X = \xi \text{ or } Y = \xi, \\ \epsilon\sqrt{\alpha} & \text{if } X, Y \in \Gamma(\mathcal{D}_h(\lambda)) \text{ or } X, Y \in \Gamma(\mathcal{D}_h(-\lambda)), \\ (\epsilon\sqrt{\alpha} - 3)\bar{g}(X, \bar{\phi}Y)^2 & \text{if } X \in \Gamma(\mathcal{D}_h(\lambda)), Y \in \Gamma(\mathcal{D}_h(-\lambda)) \text{ or } X \in \Gamma(\mathcal{D}_h(-\lambda)), Y \in \Gamma(\mathcal{D}_h(\lambda)), \end{cases}$$

where X and Y are mutually orthogonal, unit vector fields with respect to $\bar{g}$. In particular, the $\bar{\phi}$-sectional curvature is given by

$$\overline{K}(X, \bar{\phi}X) = \sqrt{\alpha} - 3$$

if $X \in \Gamma(\mathcal{D}_h(\lambda))$ or $X \in \Gamma(\mathcal{D}_h(-\lambda))$.

(ii) *If $|I_M| < 1$, the sectional curvature of the paraSasakian structure that M admits, defined as in (3.5), is:*

(3.29)
$$\widetilde{K}(X, Y) = \begin{cases} -1 & \text{if } X = \xi \text{ or } Y = \xi, \\ -\sqrt{-\alpha} & \text{if } X, Y \in \Gamma(\mathcal{D}_h(\lambda)) \text{ or } X, Y \in \Gamma(\mathcal{D}_h(-\lambda)), \\ -(\sqrt{-\alpha} - 3)\widetilde{g}(X, \widetilde{\phi}Y)^2 & \text{if } X \in \Gamma(\mathcal{D}_h(\lambda)), Y \in \Gamma(\mathcal{D}_h(-\lambda)) \text{ or } X \in \Gamma(\mathcal{D}_h(-\lambda)), Y \in \Gamma(\mathcal{D}_h(\lambda)), \end{cases}$$

where X and Y are mutually orthogonal, unit vector fields with respect to $\widetilde{g}$. In particular, the $\widetilde{\phi}$-sectional curvature is given by

$$\widetilde{K}(X, \widetilde{\phi}X) = -(\sqrt{-\alpha} - 3)$$

if $X \in \Gamma(\mathcal{D}_h(\lambda))$ *or* $X \in \Gamma(\mathcal{D}_h(-\lambda))$.

Proof. (i) By direct computation using equation (3.16).

(ii) As $\widetilde{g}$ is a semi-Riemannian metric, we first have to check that the plane fields considered in (3.29) are non-degenerate, i.e. $\widetilde{g}(X,X)\widetilde{g}(Y,Y) - \widetilde{g}(X,Y)^2 \neq 0$. Notice that for any $X \in \Gamma(\mathcal{D}_h(\lambda))$ and $Y \in \Gamma(\mathcal{D}_h(-\lambda))$,

$$\widetilde{g}(X,X) = -\frac{2-\mu+2\lambda}{\sqrt{-\alpha}}g(X,X) < 0, \qquad \widetilde{g}(Y,Y) = -\frac{2-\mu-2\lambda}{\sqrt{-\alpha}}g(Y,Y) > 0,$$

hence no vector field which is tangent to the distributions $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ is $\widetilde{g}$-isotropic. Furthermore, $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ are $\widetilde{g}$-orthogonal. Consequently one finds that all the above 2-planes $\{X,Y\}$ are non-degenerate if $X \in \Gamma(\mathcal{D}_h(\pm\lambda))$ and $Y \in \Gamma(\mathcal{D}_h(\mp\lambda))$. Whereas, if $X,Y \in \Gamma(\mathcal{D}_h(\pm\lambda))$ one has

$$\widetilde{g}(X,X)\widetilde{g}(Y,Y) - \widetilde{g}(X,Y)^2 = \left(-\frac{2-\mu\pm2\lambda}{\sqrt{-\alpha}}\right)^2 \left(g(X,X)g(Y,Y) - g(X,Y)^2\right)$$

$$\widetilde{g}(X,X)\widetilde{g}(\xi,\xi) - \widetilde{g}(X,\xi)^2 = -\frac{2-\mu\pm2\lambda}{\sqrt{-\alpha}}g(X,X)$$

so that also in these cases the 2-planes $\{X,Y\}$ and $\{X,\xi\}$ are non-degenerate. Therefore, all the above 2-planes are non-degenerate and it makes sense to compute the sectional curvature for such 2-planes. The rest follows straightforwardly from the formula (3.19). $\qquad\square$

We now discuss some consequences of Theorems 3.1 and 3.4. First notice that (3.17) implies that the metric defined in (3.4) is Sasaki-Einstein if and only if $-\epsilon n\sqrt{\alpha} + 2n + 2 = 0$, which is equivalent to requiring that $I_M > 1$ and

$$(3.30) \qquad (2-\mu+2\lambda)(2-\mu-2\lambda) = 4\left(1+\frac{1}{n}\right)^2.$$

Notice that when $I_M < -1$ the structure can never be Einstein. Analogously, by (3.20) one has that the paraSasakian metric defined in (3.5) is Einstein if and only if $n\sqrt{-\alpha} - 2n - 3 = 0$, which is equivalent to

$$(3.31) \qquad (2-\mu+2\lambda)(2-\mu-2\lambda) = -\left(2+\frac{3}{n}\right)^2.$$

In particular we deduce the following corollary.

Corollary 3.7. *The tangent sphere bundle $T_1 N$ of any space form N of constant sectional curvature $c \neq 1$ admits a canonical η-Einstein Sasakian metric $\overline{g}$ or paraSasakian metric $\widetilde{g}$, according with the circumstance that $c > 0$ or $c < 0$, respectively. The Ricci tensors of such metrics are then given by*

$$(3.32) \qquad \overline{\mathrm{Ric}} = 2(2n\sqrt{c} - 1)\overline{g} + 2(-2n\sqrt{c} + n + 1)\eta \otimes \eta,$$

$$(3.33) \qquad \widetilde{\mathrm{Ric}} = (-4n\sqrt{-c} + 3)\widetilde{g} + (4n\sqrt{-c} - 2n - 3)\eta \otimes \eta,$$

where $\dim(N) = n+1$. *Moreover, $\overline{g}$ is Einstein if and only $\dim(N) > 2$ and $c = \frac{1}{4}\left(1+\frac{1}{n}\right)^2$, and $\widetilde{g}$ is Einstein if and only if $c = -\frac{1}{16}\left(2+\frac{3}{n}\right)^2$.*

Proof. The tangent sphere bundle $T_1 N$ of a Riemannian manifold N of constant curvature $c \neq 1$ is a non-Sasakian contact metric (κ, μ)-space with $\kappa = c(2-c)$, $\mu = -2c$ (cf. § 2). Since $I_M = \frac{1+c}{|1-c|}$ one has that if $c > 0$ ($c \neq 1$) then $I_M > 1$ and if $c < 0$ then $-1 < I_M < 1$. Thus, according to the circumstance that N has positive or negative sectional curvature, respectively, we can apply (i) or (ii) of Theorems 3.1 and 3.4, and we can conclude that $T_1 N$ admits an η-Einstein Sasakian or paraSasakian metric. The expressions (3.32) and (3.33) for the corresponding Ricci tensors then follow respectively from (3.17) and (3.20), taking into account that in this case $\epsilon = 1$ and $\alpha = 16c$. Finally, the last statement easily follows, noting that one has to assume that $\dim(N) > 2$ since otherwise $c = 1$. $\qquad\square$

Remark 3.8. In [6] Blair proved that the standard contact structure on the tangent sphere bundle of a compact Riemannian manifold of nonpositive constant curvature cannot be regular. Since regularity depends only on the underlying Reeb vector field and the topological structure of the manifold, we conclude that, under the assumption of compactness of the base manifold, all the η-Einstein paraSasakian structures on tangent sphere bundles stated in Corollary 3.7 are not regular.

It is well known that in any Sasakian manifold the Reeb vector field determines a transversely Kähler foliation, i.e. a Riemannian foliation whose transverse geometry is locally modeled on a Kähler structure. Such a foliation is usually called the *Reeb foliation* and denoted $\mathcal{F}_\xi$. Since $\mathcal{F}_\xi$ is a Riemannian foliation we can use the O'Neill equation relating the Ricci tensor of the Sasakian manifold $(M, \overline{\phi}, \xi, \eta, \overline{g})$ and the Ricci tensor of the space of leaves (cf. [22, Proposition 1.7]):
(3.34)
$$\overline{\mathrm{Ric}}'(X', Y') \circ \pi = \overline{\mathrm{Ric}}(X, Y) - \frac{1}{2}\left(\overline{g}(\overline{\nabla}_X T_\xi \xi, Y) + g(X, \overline{\nabla}_Y T_\xi \xi)\right) + 2\sum_{i=1}^{2n} g(A_X X_i, A_Y X_i) + g(T_\xi X, T_\xi Y),$$

for any X', Y' local vector fields on the leaf space. Here A and T are the O'Neill tensors associated to the foliation $\mathcal{F}_\xi$ (cf. [36, p. 49]), π is a local Riemannian submersion defining $\mathcal{F}_\xi$, $\{X_1, \ldots, X_{2n}\}$ is a local orthonormal basis of the contact distribution and X, Y are the unique local vector fields tangent to the contact distribution π-related to X', Y', respectively. As $\mathcal{F}_\xi$ is totally geodesic one has that T vanishes identically. Moreover, since $A_X Y = -A_Y X = \frac{1}{2}[X, Y]_{\mathbb{R}\xi}$, (3.34) reduces to

$$\overline{\mathrm{Ric}}'(X', Y') \circ \pi = \overline{\mathrm{Ric}}(X, Y) + \frac{1}{2}\sum_{i=1}^{2n} \overline{g}(\eta([X, X_i])\xi, \eta([Y, X_i])\xi)$$

$$= \overline{\mathrm{Ric}}(X, Y) + 2\sum_{i=1}^{2n} \overline{g}(\overline{\phi}X, X_i)\overline{g}(\overline{\phi}Y, X_i)$$

$$= \overline{\mathrm{Ric}}(X, Y) + 2\overline{g}(\overline{\phi}X, \overline{\phi}Y).$$

Since $(M, \overline{\phi}, \xi, \eta, \overline{g})$ is η-Einstein with Ricci tensor given by (3.17) we conclude that

(3.35) $$\overline{\mathrm{Ric}}' = \epsilon n \sqrt{\alpha} g' = \epsilon n \sqrt{(2-\mu)^2 - 4(1-\kappa)} g',$$

where, as usual, ϵ denotes the sign of I_M. Thus we have proved that the space of leaves of any (κ, μ)-space such that $|I_M| > 1$ admits a Kähler-Einstein structure with positive scalar curvature $2n^2\sqrt{\alpha}$ if $I_M > 1$ and with negative scalar curvature $-2n^2\sqrt{\alpha}$ if $I_M < -1$. It can never be Calabi-Yau because α cannot vanish since $I_M \neq \pm 1$. This is another geometric interpretation of the sign of the Boeckx invariant of a (κ, μ)-space.

The geometric behavior of the (κ, μ)-space in the positive case seems to be very different from the negative one. In fact, when $I_M > 1$ we can apply two theorems of Tanno ([33]) getting the following result.

Corollary 3.9. *Every contact metric (κ, μ)-space $(M, \varphi, \xi, \eta, g)$ with $I_M > 1$ admits a Sasaki-Einstein metric compatible with the contact form η. Moreover, if M is compact, then its first Betti number vanishes.*

Proof. By [33, Proposition 5.3] we know that any η-Einstein K-contact metric such that a is constant and is greater than -2 can be $\mathcal{D}$-homothetically transformed in an Einstein metric. Here a is the function appearing in the η-Einstein condition (2.4). Moreover, under the further assumption that M is compact, $b_1(M) = 0$ ([33, Theorem 5.2]). Then the statements follow from Theorem 3.4 and the above remark that in our case $a > -2$ if and only if $I_M > 1$. $\square$

Remark 3.10. The topological obstruction stated in Corollary 3.9 improves the previously found restriction on the first Betti number of a (κ, μ)-space satisfying $|I_M| > 1$ that $b_1(M)$ must be an even number ([16]). More in general, a further obstruction to the existence of a Sasakian η-Einstein metric is that the first Chern class of the contact distribution must be a torsion class ([13, p. 187]). This, together with Theorem 3.4, gives another obstruction to the existence of a contact metric (κ, μ)-structure with $|I_M| > 1$.

Similar considerations can be done for the paraSasakian case, that is when $|I_M| < 1$. In such a case the Reeb foliation turns out to be transversely paraKähler and the paraSasakian metric $\widetilde{g}$ locally projects to a paraKähler-Einstein metric on the leaf space with Einstein constant $1 - n\sqrt{-\alpha}$. In particular, the Reeb foliation is transversely para-Kähler Ricci-flat if and only if $\sqrt{-\alpha} = \frac{1}{n}$. The last is a "key value" according to the following theorem.

Corollary 3.11. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ)-space such that $|I_M| < 1$. If*

$$(3.36) \qquad (2 - \mu + 2\lambda)(2 - \mu - 2\lambda) \neq -\frac{1}{n^2}$$

then M admits a paraSasakian-Einstein structure.

Proof. By [38, Theorem 4.8] we know that every paraSasakian η-Einstein manifold of dimension $2n + 1$ whose scalar curvature is different from $2n$, admits a paraSasakian-Einstein structure obtained by a $\mathcal{D}$-homothetic deformation. Since in our case, by Theorem 3.4, the scalar curvature of $(M, \widetilde{g})$ is given by $2n(2 - n\sqrt{-\alpha})$, the assertion easily follows. $\qquad \square$

Corollaries 3.7, 3.9 and 3.11 give the following consequence on tangent sphere bundles.

Corollary 3.12. *Let N be a space form of constant sectional curvature c and dimension $n + 1$.*
  (i) *If $c > 0$, $c \neq 1$ then the tangent sphere bundle $T_1 N$ carries a Sasaki-Einstein metric.*
  (ii) *If $c < 0$, $c \neq -\frac{1}{16n^2}$ then the tangent sphere bundle $T_1 N$ carries a paraSasakian-Einstein metric.*

On the other hand, for the case $I_M < -1$ we can state the following result, which is a consequence of Theorem 3.4 and [13, Corollary 24].

Corollary 3.13. *Every contact metric (κ, μ)-space with $I_M < -1$ admits a Lorentzian Sasakian-Einstein structure.*

Let us recall (cf. [4], [11]) that a *Lorentzian Sasakian* manifold is a Lorentzian manifold (M, g) of dimension $2n + 1$, endowed with a time-like vector field ξ with $g(\xi, \xi) = -1$, such that the tensor field $\phi = -\nabla\xi$ satisfies the conditions $\phi^2 X = -X - g(X, \xi)\xi$ and $(\nabla_X \phi)Y = g(Y, \xi)X - g(X, Y)\xi$. If, in addition, the Lorentzian metric g is Einstein, the manifold is said to be *Lorentzian Sasakian-Einstein*.

Another consequence in the case $I_M > 1$ deals with the notion of Weyl structure. Recall that (cf. [14]) a *Weyl structure* on a manifold M of dimension $m \geq 3$ is defined by a pair $W = ([g], D)$, where $[g]$ is a conformal class of Riemannian metrics and D is the unique torsion-free connection, called *Weyl connection*, satisfying

$$(3.37) \qquad Dg = -2\theta \otimes g.$$

for some 1-form θ. Then $(M, [g], D)$ is said to be *Einstein-Weyl* if there exists a smooth function Λ on M such that

$$(3.38) \qquad \mathrm{Ric}^D(X, Y) + \mathrm{Ric}^D(Y, X) = \Lambda g(X, Y),$$

where Ric^D denotes the Ricci tensor with respect to the connection D. Since the condition (3.37) is invariant under Weyl transformations $g' = e^{2f}g$, $\theta' = \theta + df$, with $f \in C^\infty(M)$, one sometimes abuses the terminology by choosing a Riemannian metric in $[g]$ and referring to the pair $W = (g, \theta)$ as a Weyl

structure. The existence of Einstein-Weyl structures on almost contact metric manifolds have been recently investigated by several authors ([24], [28], [29]).

Now as a consequence of Theorem 3.4 and [13, Corollary 62] we get the following result.

Corollary 3.14. *Every non-Sasakian contact (κ,μ)-space of dimension $2n+1 \geq 5$ such that $I_M > 1$ admits an Einstein-Weil structure $W = (\bar{g}', \theta)$, where $\theta = \tau\eta$, $\tau \in \mathbb{R}$, and $\bar{g}'$ is a Sasakian metric $\mathcal{D}$-homothetic to that defined by (3.4). Furthermore, if*

$$(3.39) \qquad (2-\mu)^2 - 4(1-\kappa) > 4\left(1 + \frac{1}{n}\right)^2$$

then $\bar{g}'$ belongs to the conformal class of $\bar{g}$.

Proof. In [13, Corollary 62] it was presented a necessary and sufficient condition for a K-contact manifold $(M, \varphi, \xi, \eta, g)$ of dimension greater than 3 to admit an Einstein-Weil structure $W = (g, \theta)$, with $\theta = \tau\eta$, $\tau \in \mathbb{R}$. Such condition is that M is η-Einstein with Einstein constants a, b such that $b < 0$, which by (3.17) it is easy to see occurs if and only if $\epsilon = 1$, that is $I_M > 1$, and the inequality (3.39) holds. Now let us apply a $\mathcal{D}_c$-homothetic deformation

$$\varphi' := \varphi, \quad \xi' := \frac{1}{c}\xi, \quad \eta' := c\eta, \quad g' := cg + c(c-1)\eta \otimes \eta,$$

where c is any real number chosen so that $c < \frac{\sqrt{\alpha}}{4}$. According to Remark 3.3 $(\varphi', \xi', \eta', g')$ is still a (κ', μ')-structure, with κ' and μ' given by (2.11), whose associated Sasakian structure $(\bar{\phi}', \xi', \eta', \bar{g}')$ is in turn $\mathcal{D}_c$-homothetic to the Sasakian structure $(\bar{\phi}, \xi, \eta, \bar{g})$ associated to (φ, ξ, η, g). Now, since $\alpha' = \frac{1}{c^2}\alpha$ and c was chosen in such a way that $c < \frac{\sqrt{\alpha}}{4}$ we have that

$$\sqrt{\alpha'} = \frac{1}{c}\sqrt{\alpha} > 4 > 2\left(1 + \frac{1}{n}\right).$$

Thus, according to Theorems 3.4, $\bar{g}'$ is an η'-Einstein Sasakian metric such that the second Einstein constant is negative. Consequently, we can apply [13, Corollary 62] and conclude that M admits an Einstein-Weil structure $W = (\bar{g}', \theta)$, where $\theta = \tau'\eta' = c\tau'\eta$, $\tau' \in \mathbb{R}$. Clearly, if the further assumption (3.39) is satisfied there is no need to apply a $\mathcal{D}$-homothetic deformation and the final assertion of the theorem follows. $\square$

Furthermore, by a similar reasoning to the the proof of Corollary 3.14 and by using now [13, Theorem 63] (or [24, Theorem 1]) we get the following result.

Corollary 3.15. *Every non-Sasakian contact (κ,μ)-space of dimension $2n+1 \geq 5$ such that $I_M > 1$ admits both the Einstein-Weil structure $W^+ = (\bar{g}', \theta)$ and $W^- = (\bar{g}', -\theta)$, for some 1-form θ, where $\bar{g}'$ is a Sasakian metric $\mathcal{D}$-homothetic to the canonical Sasakian metric $\bar{g}$ defined by (3.4). Moreover, if the inequality (3.39) holds then $\bar{g}'$ belongs to the conformal class of $\bar{g}$.*

Taking into account that for the tangent sphere bundle of a Riemannian manifold N of constant sectional curvature c one has $(2-\mu)^2 - 4(1-\kappa) = 16c$, we can state the following theorem.

Corollary 3.16. *Let $T_1 N$ be the tangent sphere bundle of a space form N of constant sectional curvature $c > 0$, $c \neq 1$, such that $\dim(N) = n+1 > 2$, and let $(\bar{g}, \xi, \eta, \bar{\phi})$ be the Sasakian metric on $T_1 N$ stated by Theorem 3.4, where ξ is twice the geodesic flow. Then*

(i) *$T_1 N$ carries an Einstein-Weil structure $W = (\bar{g}', \theta)$, where $\theta = \tau\eta$ for some $\tau \in \mathbb{R}$ and $\bar{g}'$ is a Sasakian metric $\mathcal{D}$-homothetic to $\bar{g}$.*

(ii) *$T_1 N$ admits both the Einstein-Weil structure $W^+ = (\bar{g}'', \theta)$ and $W^- = (\bar{g}'', -\theta)$, for some 1-form θ not necessarily proportional to η, where $\bar{g}''$ is a Sasakian metric $\mathcal{D}$-homothetic to $\bar{g}$.*

Furthermore, if the inequality

$$(3.40) \qquad c > \frac{1}{4}\left(1 + \frac{1}{n}\right)^2$$

holds then the metrics $\bar{g}'$ and $\bar{g}''$ stated in (i) and (ii), respectively, belong to the conformal class of $\bar{g}$.

We close the section with a remark concerning the case $|I_M| < 1$. By [20, Theorem 5.6] every contact metric (κ, μ)-space $(M, \varphi, \xi, \eta, g)$ such that $|I_M| < 1$ carries a canonical sequence $(\varphi_n)_{n \in \mathbb{N}}$ of contact or paracontact metric structures defined as follows:

$$(3.41) \qquad \varphi_0 := \varphi, \quad \varphi_1 := \frac{1}{2\sqrt{1-\kappa}}\mathcal{L}_\xi \varphi_0, \quad \varphi_n := \frac{1}{\sqrt{4(1-\kappa)-(2-\mu)^2}}\mathcal{L}_\xi \varphi_{n-1}, \ n \geq 2.$$

By defining

$$g_n := \begin{cases} -d\eta(\cdot, \varphi_n \cdot) + \eta \otimes \eta, & \text{if } n \text{ is even} \\ d\eta(\cdot, \varphi_n \cdot) + \eta \otimes \eta, & \text{if } n \text{ is odd} \end{cases}$$

one can prove that, for each $n \geq 1$, $(\varphi_n, \xi, \eta, g_n)$ is a contact metric (κ_n, μ_n)-structure if n is even and a paracontact metric (κ_n, μ_n)-structure if n is odd, where

$$\kappa_n = \begin{cases} \kappa + \left(1 - \frac{\mu}{2}\right)^2, & \text{if } n \text{ is even} \\ \kappa - 2 + \left(1 - \frac{\mu}{2}\right)^2, & \text{if } n \text{ is odd} \end{cases}$$

and $\mu_n = 2$. In fact, by using (2.12), one can see that $\varphi_{2n} = \varphi_2$ and $\varphi_{2n+1} = \varphi_1$ for any $n \geq 1$.

Since by assumption the Boeckx invariant of the starting contact metric (κ, μ)-structure (φ, ξ, η, g) satisfies $|I_M| < 1$, we can apply Theorems 3.1–3.4, so that M carries a canonical η-Einstein paraSasakian metric $(\widetilde{\phi}, \xi, \eta, \widetilde{g})$. On the other hand the same condition is satisfied also by the Boeckx invariant of the contact metric (κ_{2n}, μ_{2n})-structures for any $n \geq 1$, since $\mu_n = 2$. Thus, applying again Theorems 3.1–3.4, M carries an η-Einstein paraSasakian structure $(\widetilde{\phi}_{2n}, \xi, \eta, \widetilde{g}_{2n})$ for each $n \geq 1$. Actually we will prove that $(\widetilde{\phi}_{2n}, \xi, \eta, \widetilde{g}_{2n}) = (\widetilde{\phi}, \xi, \eta, \widetilde{g})$, that is the paraSasakian structure $(\widetilde{\phi}, \xi, \eta, \widetilde{g})$ stated in Theorem 3.1 is invariant under the canonical sequence (3.41). Indeed it is enough to check this for $n = 2$. We have that

$$\varphi_2 = \frac{1}{\sqrt{4(1-\kappa)-(2-\mu)^2}}\mathcal{L}_\xi \varphi_1 = \frac{1}{\sqrt{(1-\kappa)(4(1-\kappa)-(2-\mu)^2)}}\left((2-\mu)\varphi h + 2(1-\kappa)\varphi\right).$$

Then, by (3.5) and taking into account the fact that $\kappa_2 = \kappa + \left(1 - \frac{\mu}{2}\right)^2$, $\mu_2 = 2$, one has

$$(3.42) \qquad \widetilde{\phi}_2 = \frac{1}{\sqrt{4(1-\kappa_2)-(2-\mu_2)^2}}\left((2-\mu_2)\varphi_2 + 2\varphi_2 h_2\right) = \frac{1}{\sqrt{4(1-\kappa)-(2-\mu)^2}}2\varphi_2 h_2.$$

Since $h_2 = \sqrt{1 - I_M^2}h$ ([20, p. 275]), formula (3.42) becomes

$$\widetilde{\phi}_2 = \frac{2}{\sqrt{-\alpha}}\left(-\frac{\lambda(2-\mu)\sqrt{1-I_M^2}}{\sqrt{-\alpha}}\varphi^3 + \frac{2\lambda\sqrt{1-I_M^2}}{\sqrt{-\alpha}}\varphi h\right) = \frac{1}{\sqrt{-\alpha}}\left((2-\mu)\varphi + 2\varphi h\right) = \widetilde{\phi}.$$

Therefore $\widetilde{\phi}_2 = \widetilde{\phi}$ and, as the contact form is the same in both structures, we also have that $\bar{g}_2 = \bar{g}$.

4. (κ, μ)-STRUCTURES ON K-CONTACT AND K-PARACONTACT MANIFOLDS

In this section we study the converse, in some sense, of Theorem 3.1. Namely we find sufficient conditions for a Sasakian or paraSasakian manifold to admit a (κ, μ)-structure compatible with the same underlying contact form. This will also allow us to give a geometrical interpretation of the Sasakian and paraSasakian metrics defined by (3.11) and (3.13), respectively. Actually we will show that the assumption of normality is too strong, so that one can at first assume the manifold to be K-contact or K-paracontact.

Recall that a *Legendre foliation* (cf. [7]) on a contact manifold (M,η) is nothing but an integrable n-dimensional subbundle of the contact distribution. Legendre foliations have been extensively investigated in recent years from various points of views. In particular Pang ([30]) provided a classification of Legendre foliations by means of a bilinear symmetric form $\Pi_{\mathcal{F}}$ on the tangent bundle of the foliation $\mathcal{F}$, defined by

$$\Pi_{\mathcal{F}}(X,X') = -\left(\mathcal{L}_X \mathcal{L}_{X'}\eta\right)(\xi) = 2d\eta([\xi,X],X').$$

He called a Legendre foliation *non-degenerate*, *degenerate* or *flat* according to the circumstance that the bilinear form $\Pi_{\mathcal{F}}$ is non-degenerate, degenerate or vanishes identically, respectively. For a non-degenerate Legendre foliation $\mathcal{F}$, Libermann ([27]) defined also a linear map $\Lambda_{\mathcal{F}} : TM \longrightarrow T\mathcal{F}$, whose kernel is $T\mathcal{F} \oplus \mathbb{R}\xi$, such that

$$(4.1) \qquad\qquad \Pi_{\mathcal{F}}(\Lambda_{\mathcal{F}}Z, X) = d\eta(Z,X)$$

for any $Z \in \Gamma(TM)$, $X \in \Gamma(T\mathcal{F})$. The operator $\Lambda_{\mathcal{F}}$ is surjective and satisfies $(\Lambda_{\mathcal{F}})^2 = 0$ and

$$(4.2) \qquad\qquad \Lambda_{\mathcal{F}}[\xi,X] = \frac{1}{2}X$$

for all $X \in \Gamma(T\mathcal{F})$. Then one can extend $\Pi_{\mathcal{F}}$ to a symmetric bilinear form defined on all TM by setting

$$\overline{\Pi}_{\mathcal{F}}(Z,Z') := \begin{cases} \Pi_{\mathcal{F}}(Z,Z') & \text{if } Z,Z' \in \Gamma(T\mathcal{F}) \\ \Pi_{\mathcal{F}}(\Lambda_{\mathcal{F}}Z, \Lambda_{\mathcal{F}}Z') & \text{otherwise.} \end{cases}$$

Now let (M,φ,ξ,η,g) be a non-Sasakian contact metric (κ,μ)-space. Being n-dimensional integrable subbundles of the contact distribution, the eigenspaces of h, $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$, define two mutually orthogonal Legendre foliations on M, where we recall that $\lambda = \sqrt{1-\kappa}$. Such a foliated structure of a (κ,μ)-space was studied in [16]. In particular, more explicit formulas for the Pang invariants of $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ were found, namely

$$(4.3) \qquad \Pi_{\mathcal{D}_h(\lambda)} = (2-\mu+2\lambda)g|_{\mathcal{D}_h(\lambda)\times\mathcal{D}_h(\lambda)}, \quad \Pi_{\mathcal{D}_h(-\lambda)} = (2-\mu-2\lambda)g|_{\mathcal{D}_h(-\lambda)\times\mathcal{D}_h(-\lambda)},$$

from which it follows that $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ are both non-degenerate if and only if $I_M \neq \pm 1$.

Let us assume that $|I_M| > 1$. By using (4.3) and (3.11) one has, for all $X,X' \in \Gamma(\mathcal{D}_h(\lambda))$,

$$\begin{aligned} \Pi_{\mathcal{D}_h(\lambda)}(X,X') &= (2-\mu)g(X,X') + 2g(\lambda X, X') \\ &= (2-\mu)g(X,X') + 2g(hX, X') \\ &= \epsilon\sqrt{\alpha}\,\bar{g}(X,X'), \end{aligned}$$

where $\alpha = (2-\mu)^2 - 4(1-\kappa)$ and ϵ is the sign of I_M, according to the notation of § 3. Analogously, for all $Y,Y' \in \Gamma(\mathcal{D}_h(-\lambda))$:

$$\begin{aligned} \Pi_{\mathcal{D}_h(-\lambda)}(Y,Y') &= (2-\mu)g(Y,Y') + 2g(-\lambda Y, Y') \\ &= (2-\mu)g(Y,Y') + 2g(hY, Y') \\ &= \epsilon\sqrt{\alpha}\,\bar{g}(Y,Y'). \end{aligned}$$

Whereas, if $|I_M| < 1$ one has

$$\Pi_{\mathcal{D}_h(\lambda)}(X,X') = -\sqrt{-\alpha}\,\widetilde{g}(X,X'), \quad \Pi_{\mathcal{D}_h(\lambda)}(Y,Y') = -\sqrt{-\alpha}\,\widetilde{g}(Y,Y').$$

Therefore, geometrically the Sasakian metric $\bar{g}$ in the case $|I_M| > 1$ and the paraSasakian metric $\widetilde{g}$ in the case $|I_M| < 1$ represent, up to a constant factor, the Pang invariant of the foliations $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ when restricted to the leaves. Moreover, notice that since $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ are totally geodesic foliations ([8, Proposition 3.7]) and because of (3.15) and (3.18), they are still totally geodesic with respect to $\bar{g}$ and $\widetilde{g}$.

Now we prove that the above properties determine uniquely the contact metric (κ,μ)-structure.

Theorem 4.1. *Let $(M, \bar{\phi}, \xi, \eta, \bar{g})$ be a K-contact manifold endowed with two totally geodesic, mutually orthogonal Legendre foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. Assume that*

$$(4.4) \qquad \Pi_{\mathcal{F}_1} = \epsilon\sqrt{ab}\,\bar{g}|_{\mathcal{F}_1 \times \mathcal{F}_1}, \quad \Pi_{\mathcal{F}_2} = \epsilon\sqrt{ab}\,\bar{g}|_{\mathcal{F}_2 \times \mathcal{F}_2}$$

for some real numbers a and b such that $a \neq b$ and $a \cdot b > 0$, where

$$\epsilon = \begin{cases} 1, & \text{if } a > 0, \, b > 0 \\ -1, & \text{if } a < 0, \, b < 0. \end{cases}$$

Then M admits a contact metric $(\kappa_{a,b}, \mu_{a,b})$-structure $(\varphi_{a,b}, \xi, \eta, g_{a,b})$, compatible with the original contact form η, where

$$(4.5) \qquad \kappa_{a,b} = 1 - \frac{(a-b)^2}{16}, \quad \mu_{a,b} = 2 - \frac{a+b}{2}.$$

Furthermore, $(M, \bar{\phi}, \xi, \eta, \bar{g})$ is Sasakian and η-Einstein with Ricci tensor

$$(4.6) \qquad \overline{\mathrm{Ric}} = (\epsilon n\sqrt{ab} - 2)\bar{g} + (-\epsilon n\sqrt{ab} + 2n + 2)\eta \otimes \eta.$$

Proof. First notice that, since $\mathcal{F}_1$ and $\mathcal{F}_2$ are mutually $\bar{g}$-orthogonal, $\bar{\phi}T\mathcal{F}_1 \subset T\mathcal{F}_2$ and $\bar{\phi}T\mathcal{F}_2 \subset T\mathcal{F}_1$. Indeed let us fix $X \in \Gamma(T\mathcal{F}_1)$. Then for any $X' \in \Gamma(T\mathcal{F}_1)$ one has $\bar{g}(\bar{\phi}X, X') = -d\eta(X, X') = \frac{1}{2}\eta([X, X']) = 0$ since $\mathcal{F}_1$ is a Legendre foliation. Moreover, obviously $\bar{g}(\bar{\phi}X, \xi) = 0$. Consequently, since by assumption $(T\mathcal{F}_1 \oplus \mathbb{R}\xi)^\perp = T\mathcal{F}_2$, we conclude that $\bar{\phi}X \in \Gamma(T\mathcal{F}_2)$. In a similar way one can prove the other relation. Now, taking into account the decomposition of the tangent bundle of M as $TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2 \oplus \mathbb{R}\xi$, let us define a $(1,1)$-tensor field $\varphi_{a,b}$ and a Riemannian metric $g_{a,b}$ on M by

$$(4.7) \qquad \varphi_{a,b} := \begin{cases} \sqrt{\frac{b}{a}}\,\bar{\phi}, & \text{on } T\mathcal{F}_1 \\ \sqrt{\frac{a}{b}}\,\bar{\phi}, & \text{on } T\mathcal{F}_2 \\ 0, & \text{on } \mathbb{R}\xi \end{cases} \qquad g_{a,b} := \begin{cases} \sqrt{\frac{b}{a}}\,\bar{g}, & \text{on } T\mathcal{F}_1 \times T\mathcal{F}_1 \\ \sqrt{\frac{a}{b}}\,\bar{g}, & \text{on } T\mathcal{F}_2 \times T\mathcal{F}_2 \\ \eta \otimes \eta, & \text{otherwise} \end{cases}$$

and extend it by linearity. A straightforward computation shows that $\varphi_{a,b}^2 = -I + \eta \otimes \xi$ and $g_{a,b}(\varphi_{a,b}\cdot, \varphi_{a,b}\cdot) = g_{a,b} - \eta \otimes \eta$. Let us check that $(\varphi_{a,b}, \xi, \eta, g_{a,b})$ is a contact metric structure, i.e.

$$(4.8) \qquad d\eta(X, Y) = g_{a,b}(X, \varphi_{a,b}Y)$$

for all $X, Y \in \Gamma(TM)$. If either X or Y belongs to $\mathbb{R}\xi$, (4.8) is trivially satisfied, both sides being zero. Let $X, Y \in \Gamma(T\mathcal{F}_1)$. Then $d\eta(X, Y) = -\frac{1}{2}\eta([X, Y]) = 0$ because $\mathcal{F}_1$ is a Legendre foliation. On the other hand, since $\bar{\phi}Y \in \Gamma(T\mathcal{F}_2)$, then $g_{a,b}(X, \varphi_{a,b}Y) = \sqrt{\frac{b}{a}}g_{a,b}(X, \bar{\phi}Y) = \sqrt{\frac{b}{a}}\eta(X)\eta(\bar{\phi}Y) = 0 = d\eta(X, Y)$. Analogously, if $X, Y \in \Gamma(T\mathcal{F}_2)$, then $g_{a,b}(X, \bar{\phi}Y) = 0 = d\eta(X, Y)$. It remains the case when $X \in \Gamma(T\mathcal{F}_1)$ and $Y \in \Gamma(T\mathcal{F}_2)$. In that case by using (4.7) one has $g_{a,b}(X, \varphi_{a,b}Y) = \sqrt{\frac{a}{b}}\sqrt{\frac{b}{a}}\bar{g}(X, \bar{\phi}Y) = \bar{g}(X, \bar{\phi}Y) = d\eta(X, Y)$, since $(\bar{\phi}, \xi, \eta, \bar{g})$ is a contact metric structure. Notice that, directly by the definition of $g_{a,b}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are $g_{a,b}$-orthogonal, so that the tensor field $\varphi_{a,b}$ also maps $T\mathcal{F}_1$ on $T\mathcal{F}_2$ and $T\mathcal{F}_2$ on $T\mathcal{F}_1$. Finally we prove that $(M, \varphi_{a,b}, \xi, \eta, g_{a,b})$ is a (κ, μ)-space. First of all, we compute the operator $h_{a,b}$ of

the contact metric structure $(\varphi_{a,b}, \xi, \eta, g_{a,b})$. We have, for any $X \in \Gamma(T\mathcal{F}_1)$,

$$
\begin{aligned}
2h_{a,b}X &= [\xi, \varphi_{a,b}X] - \varphi_{a,b}[\xi, X] \\
&= [\xi, \varphi_{a,b}X] - \varphi_{a,b}([\xi, X]_{\mathcal{F}_1}) - \varphi_{a,b}([\xi, X]_{\mathcal{F}_2}) \\
&= \sqrt{\frac{b}{a}}[\xi, \bar{\phi}X] - \sqrt{\frac{b}{a}}\bar{\phi}([\xi, X]_{\mathcal{F}_1}) - \sqrt{\frac{a}{b}}\bar{\phi}([\xi, X]_{\mathcal{F}_2}) \\
&= \sqrt{\frac{b}{a}}[\xi, \bar{\phi}X]_{\mathcal{F}_1} + \sqrt{\frac{b}{a}}[\xi, \bar{\phi}X]_{\mathcal{F}_2} - \sqrt{\frac{b}{a}}\bar{\phi}([\xi, X]_{\mathcal{F}_1}) - \sqrt{\frac{a}{b}}\bar{\phi}([\xi, X]_{\mathcal{F}_2}) \\
&= \sqrt{\frac{b}{a}}[\xi, \bar{\phi}X]_{\mathcal{F}_1} + 2\sqrt{\frac{b}{a}}(\bar{h}X)_{\mathcal{F}_2} - \sqrt{\frac{a}{b}}\bar{\phi}([\xi, X]_{\mathcal{F}_2}) \\
&= \left(\sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}}\right)(\bar{\phi}[\xi, X])_{\mathcal{F}_1},
\end{aligned}
\tag{4.9}
$$

where we have used the K-contact condition $\bar{h} = 0$ and we have decomposed the vector field $[\xi, X]$ according to the decomposition $TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2 \oplus \mathbb{R}\xi$. In particular, from (4.9) it follows that $h_{a,b}$ maps $T\mathcal{F}_1$ on $T\mathcal{F}_1$. Now we use the assumption (4.4) for finding a more explicit expression for the Liberman map $\Lambda_{\mathcal{F}_1} : TM \longrightarrow T\mathcal{F}_1$. By using (4.1) we have, for any $X \in \Gamma(T\mathcal{F}_1)$ and $Z \in \Gamma(TM)$, that

$$
\epsilon\sqrt{ab}\,\bar{g}(\Lambda_{\mathcal{F}_1}Z, X) = \Pi_{\mathcal{F}_1}(\Lambda_{\mathcal{F}_1}Z, X) = d\eta(Z, X) = \bar{g}(Z, \bar{\phi}X) = -\bar{g}(X, \bar{\phi}Z),
$$

from which it follows that

$$
\Lambda_{\mathcal{F}_1}Z = -\epsilon\frac{1}{\sqrt{ab}}(\bar{\phi}Z)_{\mathcal{F}_1}.
\tag{4.10}
$$

Then, by using (4.2) and (4.10), we get

$$
(\bar{\phi}[\xi, X])_{\mathcal{F}_1} = -\epsilon\sqrt{ab}\,\Lambda_{\mathcal{F}_1}[\xi, X] = -\epsilon\frac{\sqrt{ab}}{2}X.
\tag{4.11}
$$

Therefore, due to (4.11), formula (4.9) becomes

$$
h_{a,b}X = -\epsilon\frac{\sqrt{ab}}{4}\left(\sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}}\right)X = \epsilon\frac{|a| - |b|}{4}X = \frac{a - b}{4}X,
\tag{4.12}
$$

since $\epsilon|a| = a$ and $\epsilon|b| = b$. Due to $h_{a,b}\varphi_{a,b} = -\varphi_{a,b}h_{a,b}$, from (4.12) it follows that

$$
h_{a,b}Y = -\frac{a - b}{4}Y
\tag{4.13}
$$

for any $Y \in \Gamma(T\mathcal{F}_2)$. Thus we have proved that the tangent bundles of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ coincide with the eigendistributions of $h_{a,b}$ corresponding to the constant eigenvalues $\pm\lambda_{a,b}$, where $\lambda_{a,b} := \frac{|a-b|}{4}$. Let ∇^{bl} be the bi-Legendrian connection associated to $(\mathcal{F}_1, \mathcal{F}_2)$ (cf. [17]). By definition ∇^{bl} is the unique linear connection on M such that the following conditions hold:

   (i) $\nabla^{bl}\mathcal{F}_1 \subset \mathcal{F}_1$, $\nabla^{bl}\mathcal{F}_2 \subset \mathcal{F}_2$
   (ii) $\nabla^{bl}\eta = 0$, $\nabla^{bl}d\eta = 0$,
   (iii) $T^{bl}(X, Y) = 2d\eta(X, Y)\xi$ for any $X \in \Gamma(T\mathcal{F}_1)$, $Y \in \Gamma(T\mathcal{F}_2)$,
        $T^{bl}(Z, \xi) = [\xi, Z_{\mathcal{F}_1}]_{\mathcal{F}_2} + [\xi, Z_{\mathcal{F}_2}]_{\mathcal{F}_1}$ for any $Z \in \Gamma(TM)$,

where T^{bl} denotes the torsion tensor field of ∇^{bl}. Actually, because of the integrability of $T\mathcal{F}_1$ and $T\mathcal{F}_2$, one has immediately that $T^{bl}(X, X') = 0 = 2d\eta(X, X')\xi$ for any $X, X' \in \Gamma(T\mathcal{F}_1)$ and $T^{bl}(Y, Y') = 0 = 2d\eta(Y, Y')\xi$ for any $Y, Y' \in \Gamma(T\mathcal{F}_2)$. Thus the first identity in (iii) holds for any two vector fields on the contact distribution. Moreover, since $\mathcal{F}_1$ and $\mathcal{F}_2$ are assumed to be totally geodesic with respect to

$\bar{g}$, due to [17, Proposition 2.9], we get that $\nabla^{bl}\bar{\phi} = 0$. Then, because of (i) and (4.7), one has that ∇^{bl} preserves also $\varphi_{a,b}$. Next, as $d\eta = g_{a,b}(\cdot, \varphi_{a,b}\cdot)$, after a straightforward computation one obtains

$$(\nabla^{bl}_Z g_{a,b})(Z', Z'') = -(\nabla^{bl}_Z d\eta)(Z', \varphi_{a,b}Z'') - d\eta(Z', (\nabla^{bl}_Z \varphi_{a,b})Z'') + \eta(Z'')(\nabla^{bl}_Z \eta)(Z') + \eta(Z')(\nabla^{bl}_Z \eta)(Z'')$$

for all $Z, Z', Z'' \in \Gamma(TM)$. Since ∇^{bl} preserves η, $d\eta$ and $\varphi_{a,b}$, the above equation implies that $\nabla^{bl}g_{a,b} = 0$. Finally (i) and (4.12)–(4.13) easily ensure that ∇^{bl} preserves also the tensor field $h_{a,b}$. Therefore the bi-Legendrian connection ∇^{bl} defined by the foliations $\mathcal{F}_1$, $\mathcal{F}_2$ satisfies all the conditions required by [19, Theorem 4.4] and we can conclude that $(\varphi_{a,b}, \xi, \eta, g_{a,b})$ is a (κ, μ)-structure. In order to find explicitly the constants κ and μ (which of course will depend on a and b) we notice that, since $\Pi_{\mathcal{F}_1}$ is an invariant of the Legendre foliation $\mathcal{F}_1$, the two expressions for $\Pi_{\mathcal{F}_1}$ in (4.3) and (4.4) have to coincide. We thus have, for any $0 \neq X \in \Gamma(T\mathcal{F}_1)$,

$$(2 - \mu_{a,b} + 2\lambda_{a,b})g_{a,b}(X, X) = \epsilon\sqrt{ab}\bar{g}(X, X) = \epsilon|a|g_{a,b}(X, X) = ag_{a,b}(X, X).$$

Thus

(4.14) $$2 - \mu_{a,b} + 2\lambda_{a,b} = a.$$

Arguing in a similar way for $\mathcal{F}_2$ one gets

(4.15) $$2 - \mu_{a,b} - 2\lambda_{a,b} = b.$$

Then by using (4.14)–(4.15) we get the values (4.5) for $\kappa_{a,b}$ and $\mu_{a,b}$. It remains to prove the last part of the statement. From (4.7) and (4.14)–(4.15) one has immediately that for any $X, X' \in \Gamma(T\mathcal{F}_1)$

$$\begin{aligned}
\bar{g}(X, X') &= \sqrt{\frac{a}{b}}g_{a,b}(X, X')\\
&= \frac{a}{\sqrt{ab}}g_{a,b}(X, X')\\
&= \frac{1}{\sqrt{(2 - \mu_{a,b})^2 - 4(1 - \kappa_{a,b})}}\left((2 - \mu_{a,b})g_{a,b}(X, X') + 2g_{a,b}(\lambda_{a,b}X, X')\right)\\
&= \frac{1}{\sqrt{(2 - \mu_{a,b})^2 - 4(1 - \kappa_{a,b})}}\left((2 - \mu_{a,b})g_{a,b}(X, X') + 2g_{a,b}(h_{a,b}X, X')\right).
\end{aligned}$$

Similarly, one has that

$$\bar{g}(Y, Y') = \frac{1}{\sqrt{(2 - \mu_{a,b})^2 - 4(1 - \kappa_{a,b})}}\left((2 - \mu_{a,b})g_{a,b}(Y, Y') + 2g_{a,b}(h_{a,b}Y, Y')\right)$$

for any $Y, Y' \in \Gamma(T\mathcal{F}_2)$. Consequently, as $T\mathcal{F}_1 = \mathcal{D}_{h_{a,b}}(\lambda_{a,b})$ and $T\mathcal{F}_2 = \mathcal{D}_{h_{a,b}}(-\lambda_{a,b})$, we see that the metric $\bar{g}$ coincides with the Sasakian metric defined by (3.11). Then $\bar{g}$ is also η-Einstein with Ricci tensor (4.6) because of Theorem 3.4. □

Notice that each contact metric $(\kappa_{a,b}, \mu_{a,b})$-structure $(\varphi_{a,b}, \xi, \eta, g_{a,b})$ of the family stated in Theorem 4.1 has a Boeckx invariant whose absolute value is strictly greater than 1. Indeed from (4.5) it follows that

$$I_M = \frac{a + b}{|a - b|}.$$

Finally, we will examine the paracontact case.

Theorem 4.2. *Let $(M, \widetilde{\phi}, \xi, \eta, \widetilde{g})$ be a K-paracontact manifold endowed with two totally geodesic, mutually orthogonal Legendre foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, such that $\widetilde{g}|_{\mathcal{F}_1 \times \mathcal{F}_1}$ is positive definite and $\widetilde{g}|_{\mathcal{F}_2 \times \mathcal{F}_2}$ is negative definite. If*

(4.16) $$\Pi_{\mathcal{F}_1} = -\sqrt{-ab}\widetilde{g}|_{\mathcal{F}_1 \times \mathcal{F}_1}, \quad \Pi_{\mathcal{F}_2} = -\sqrt{-ab}\widetilde{g}|_{\mathcal{F}_2 \times \mathcal{F}_2}$$

for some real numbers a and b such that $a \neq b$ and $a \cdot b < 0$, then M admits a contact metric (κ, μ)-structure $(\varphi_{a,b}, \xi, \eta, g_{a,b})$, compatible with the original contact form η, where

$$(4.17) \qquad \kappa = 1 - \frac{(a-b)^2}{16}, \quad \mu = 2 - \frac{a+b}{2}.$$

Furthermore, $(M, \bar{\phi}, \xi, \eta, \bar{g})$ is paraSasakian and η-Einstein with Ricci tensor

$$\widetilde{\mathrm{Ric}} = (-n\sqrt{-ab} + 3)\widetilde{g} + (n\sqrt{-ab} - 2n - 3)\eta \otimes \eta.$$

Proof. The proof is very similar to that of Theorem 4.1, the only difference is the definition of the structure tensors $\varphi_{a,b}$ and $g_{a,b}$, which now are given by

$$(4.18) \qquad \varphi_{a,b} := \begin{cases} \frac{\sqrt{-ab}}{a}\widetilde{\phi}, & \text{on } T\mathcal{F}_1 \\ \frac{\sqrt{-ab}}{b}\widetilde{\phi}, & \text{on } T\mathcal{F}_2 \\ 0, & \text{on } \mathbb{R}\xi \end{cases} \qquad g_{a,b} := \begin{cases} -\frac{\sqrt{-ab}}{a}\widetilde{g}, & \text{on } T\mathcal{F}_1 \times T\mathcal{F}_1 \\ -\frac{\sqrt{-ab}}{b}\widetilde{g}, & \text{on } T\mathcal{F}_2 \times T\mathcal{F}_2 \\ \eta \otimes \eta, & \text{otherwise} \end{cases}$$

The extra-assumption that $\widetilde{g}|_{\mathcal{F}_1 \times \mathcal{F}_1}$ is positive definite and $\widetilde{g}|_{\mathcal{F}_2 \times \mathcal{F}_2}$ is negative definite ensure that the symmetric tensor $g_{a,b}$ defined by (4.18) is a Riemannian metric. The rest of the proof goes as in Theorem 4.1, once one notices that [17, Proposition 2.9], which is used for proving that (φ, ξ, η, g) is a (κ, μ)-structure, straightforwardly holds also in the context of paracontact metric geometry. $\qquad\square$

REFERENCES

[1] M. T. K. Abbassi, O. Kowalski, *On Einstein Riemannian g-natural metrics on unit tangent sphere bundles*, Ann. Glob. Anal. Geom. **38** (2010), 1120.

[2] P. Alegre, A. Carriazo, *Structures on generalized Sasakian-space-forms*, Differential Geom. Appl. **26** (2008), no. 6, 656–666.

[3] D. V.Alekseevsky, V. Cortés, A. Galaev, T. Leistner, *Cones over pseudo-Riemannian manifolds and their holonomy*, J. Reine Angew. Math. **635** (2009), 23-69.

[4] H. Baum, *Twistor and Killing spinors in Lorentzian geometry*, Global analysis and harmonic analysis (Marseille-Luminy, 1999), Semin. Congr., **4**, Paris: Soc. Math. France, 2000, pp. 3552.

[5] D. E Blair, *Two remarks on contact metric structures*, Tôhoku Math. J. **28** (1976), 373–379.

[6] D. E. Blair, *On the non-regularity of tangent sphere bundles*, Proc. Royal Soc. Edinburgh **82** (1978), 13–17.

[7] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds, 2nd Edition*, Progress in Mathematics **203**, Birkhäuser, Boston, 2010.

[8] D. E. Blair, T. Koufogiorgos, B. J. Papantoniou, *Contact metric manifolds satisfyng a nullity condition*, Israel J. Math. **91** (1995), 189–214.

[9] E. Boeckx, *A full classification of contact metric (κ, μ)-spaces*, Illinois J. Math. **44** (2000), 212–219.

[10] E. Boeckx, L. Vanhecke, *Unit tangent sphere bundles with constant scalar curvature*, Czechoslovak Math. J. **51** (2001), 523-544.

[11] C. Bohle, *Killing spinors on Lorentzian manifolds*, J. Geom. Phys. **45** (2003), 285-308.

[12] C. P. Boyer, K. Galicki, *Sasakian geometry*, Oxford University Press, 2008.

[13] C. P. Boyer, K. Galicki, P. Matzeu, *On η-Einstein Sasakian geometry*, Commun. Math. Phys. **262** (2006), 177–208.

[14] D. M. J. Calderbank, H. Pedersen, *Einstein-Weyl geometry*, Surveys in differential geometry: essays on Einstein manifolds. Surv. Differ. Geom., VI, Boston, MA: Int. Press, 1999, pp. 387-423

[15] B. Cappelletti Montano, *Bi-Legendrian structures and paracontact geometry*, Int. J. Geom. Met. Mod. Phys. **6** (2009), 487–504.

[16] B. Cappelletti Montano, *The foliated structure of contact metric (κ, μ)-spaces*, Illinois J. Math. **53** (2009), 1157–1172.

[17] B. Cappelletti Montano, *Some remarks on the generalized Tanaka-Webster connection of a contact metric manifold*, Rocky Mountain J. Math. **40** (2010), 1009–1037.

[18] B. Cappelletti Montano, *Bi-paracontact structures and Legendre foliations*, Kodai Math. J. **33** (2010), 473–512.

[19] B. Cappelletti Montano, L. Di Terlizzi, *Contact metric (κ, μ)-spaces as bi-Legendrian manifolds*, Bull. Austral. Math. Soc. **77** (2008), 373–386.

[20] B. Cappelletti Montano, L. Di Terlizzi, *Geometric structures associated with a contact metric (κ, μ)-space*, Pacific J. Math. **246** (2010), 257–292.

[21] Y. D. Chai, S. H. Chun, J. H. Park, K. Sekigawa, *Remarks on η-Einstein unit tangent bundles*, Monatsh. Math. **155** (2008), 31–42.

[22] M. Falcitelli, S. Ianus, A. M. Pastore, *Riemannian submersions and related topics*, World Scientific, Singapore, 2004.

[23] R. R. Gomez, *Lorentzian Sasaki-Einstein metrics on connected sums of $S^2 \times S^3$*, Geom. Dedicata **150** (2011), 249–255.

[24] A. Gosh, *EinsteinWeyl structures on contact metric manifolds*, Ann. Glob. Anal. Geom. **35** (2009), 431-441.

[25] S. Ivanov, D. Vassilev, S. Zamkovoy, *Conformal paracontact curvature and the local flatness theorem*, Geom. Dedicata **144** (2010), 79-100.

[26] S. Kaneyuki, F. L. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J. **99** (1985), 173–187.

[27] P. Libermann, *Legendre foliations on contact manifolds*, Differential Geom. Appl. **1** (1991), 57–76.

[28] P. Matzeu, *Closed EinsteinWeyl structures on compact Sasakian and cosymplectic manifolds*, Proc. Edinb. Math. Soc. **54** (2011), 149–160.

[29] F. Narita, *Riemannian submersions and Riemannian manifolds with Einstein-Weyl structures*, Geom. Dedicata **65** no. 1 (1997) 103-116.

[30] M. Y. Pang, *The structure of Legendre foliations*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 417–453.

[31] J. H. Park, K. Sekigawa, *When are the tangent sphere bundles of a Riemannian manifold η-Einstein?*, Ann. Glob. Anal. Geom. **36** (2009), 275–284.

[32] D. Perrone, *Contact metric manifolds whose characteristic vector field is a harmonic vector field*, Differential Geom. Appl. **20** (2004), no. 3, 367–378.

[33] S. Tanno, *Harmonic forms and Betti numbers of certain contact Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 308–316.

[34] S. Tanno, *Geodesic flows on CL-manifolds and Einstein metrics on $S^3 \times S^2$*, Minimal Submanifolds and Geodesics, Kaigai, Tokyo, 1978, 283-292.

[35] Tanno S., *Variational problems on contact manifolds*, Trans. Amer. Math. Soc. **314** (1989), 349–379.

[36] P. Tondeur, *Geometry of foliations*, Monographs in Mathematics **90**, Birkhäuser, Boston, 1997.

[37] E. Vergara-Diaz, C. M. Wood, *Harmonic contact metric structures*, Geom. Dedicata **123** (2006), 131–151.

[38] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Glob. Anal. Geom. **36** (2009), 37–60.

[39] S. Zamkovoy, D. Tzanov, *Non-existence of flat paracontact metric structures in dimension greater than or equal to five*, preprint (available on arXiv:0910.5838v1).

Dipartimento di Matematica e Informatica, Università degli Studi di Cagliari, Via Ospedale 72, 09124 Cagliari, ITALY

E-mail address: `b.cappellettimontano@gmail.com`

Departamento de Geometría y Topología, Universidad de Sevilla, Aptdo. de Correos 1160, 41080 Sevilla, SPAIN

E-mail address: `carriazo@us.es`

Departamento de Geometría y Topología, Universidad de Sevilla, Aptdo. de Correos 1160, 41080 Sevilla, SPAIN

E-mail address: `veronicamartin@us.es`